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MODELLING SYSTEMS OF INTERCONNECTED LAKES

by

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Abstract

A model for determining the interaction between the lakes in a lake system is derived. The continuity equations for all the lakes in a lake system are coupled through discharge equations based on measured resistance factors for the channels interconnecting the lakes. A resistance factor shown to be a unique function of the mean of the up- and downstream lake levels is introduced. The proposed model is applied to a system of six lakes, which are also interconnected with the sea. Continuous simulation of the levels is performed and comparison is made with observed lake levels.

1. *Introduction*

In some lake systems consisting of many lakes or ponds interconnected by short river reaches or just ditches the water can flow from one lake to another in either direction. A lake can also be connected to the sea and depending on sea level fluctuations and runoff into the lake the water exchange between the sea and the lake can take place in either direction. It is clear lake routing cannot be performed for each lake separately, since it is not known which lake that is the upstream lake and since the different lakes affect each other. In this paper a model is proposed by which lake levels and water exchange between different lakes in a lake system can be determined. The water passages between different lakes are treated as point losses of energy head. These point losses give a relation between discharge and water level difference between two lakes. In this way the continuity equations for the lakes are coupled. First the mathematical background

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of the model is shown. Thereafter a discussion is made concerning the accuracy of considering the water passages as point losses. The model is applied to a lake system consisting of six lakes and comparison is made with independent observations.

Studies of and modelling of water exchange through narrow straits between the sea or a large lake and an enclosed bay or estuary have been rather frequently reported in the literature. For these situations density stratification and wind induced currents must be considered, or the flow must at least be treated as two-layer open channel flow. Simple gravity exchange flow for tidal flows in inlets has been studied analytically, *e.g.* by BAINES (1958) and KEULEGAN (1967). A method not using harmonic forcing functions was used by DICK and MARSALEK (1973) for calculating the gravity exchange flow through the Burlington Canal between Lake Ontario and Hamilton Harbour. Burlington Canal is a short 10 m deep canal built for large ships. In the present study the conditions are very different. Systems of many interconnected lakes are considered. The channels between the lakes are shallow streams in which the flow can be in either direction, and where the resistance to flow is high. When lake levels are to be computed all the lakes must be considered at the same time. In the applications there are no density differences between the different lake waters or the water of the downstream large water body. Therefore density stratification does not need to be considered.

2. *The mathematical model*

The continuity equation for a single lake, i, is

$$\frac{\partial S_i}{\partial t} = Q_{ei} + Q_{i-1,i} - Q_{i,i+1} \tag{1}$$

where S = lake water storage above a datum for indexed lake, t = time, Q_e = external inflow, *i.e.* runoff from an upstream catchment into lake of second index, $Q_{i-1,i}$ = discharge from lake $i-1$ to lake i, $Q_{i,i+1}$ = discharge from lake i to lake $i+1$. For numerical computations eq. (1) is written on an explicit finite difference form.

The flow between two lakes depends on the characteristics of the stream between the two lakes, *i.e.* friction factor, length of reach, cross section area and shape, and on the difference in total energy head between the two lakes. If the reach is not very long and the lake level fluctuations are slow, the flow in the channel between the lakes can be considered to be a steady-state flow. Knowing the lake levels the discharge between two lakes can be determined from non-uniform flow computations. Neglecting acceleration losses at the outlet from the upper lake, the difference in potential energy between the lakes can be written as

$$dh = \int_0^L S_f dx \tag{2}$$

where dh = difference in potential energy between two lakes, L = length of channel interconnecting the two lakes, x = distance coordinate along the channel, S_f = friction slope along the channel.

The friction slope determined by Manning's equation can be substituted into eq. (2). Then, the new equation

$$dh = Q^2 \int_0^L n^2 R^{-4/3} A^2 dx \tag{3}$$

where n = Manning's n, A = cross section area, R = hydraulic radius, can be re-arranged so that the discharge is found as

$$Q = dh^{\frac{1}{2}} f^{-1} \tag{4}$$

where f may be called a »resistance factor», which is for the general situation

$$f^2 = \int_0^L n^2 A^{-2} R^{-4/3} dx \tag{5}$$

and for the situation of a uniform channel and only small changes in water depth along the reach

$$f = n L^{1/2} A^{-1} R^{-2/3} \tag{6}$$

From measurements of discharge and difference in lake levels the resistance factor can be found from eq. (4). This equation is valid as long as the flow is turbulent, even if Manning's formula is not applicable. The resistance factor should be a function of some average channel depth, but for large relative changes of water depth over a reach also the difference in water level between two lakes influences the resistance factor. In the next section it is investigated, when f can be considered as a unique function of depth, and which errors are involved, when gradually varied flow computations are replaced by point loss considerations.

For a lake system interconnected with a large water body, whose stage fluctuations are known, and where the lake levels are influenced by the stage of the large water body, eqs. (1) and (4) are used to determine lake levels and flow between the lakes. First, eq. (4) is used to determine the flow in each channel. Then, knowing the net inflow to each lake, including runoff from upstream catchments, an explicit finite difference form of eq. (1) is used to determine the lake level of each lake one time step forward. The procedure is repeated until a predetermined time is reached.

Apart from knowing f as a function of depth for each channel the lake surface area as a function of water level above a datum must be known for all the lakes. Even if the external inflows and the known stage of the large water body do not change much over a day, a time step of more than one or two hours should not be used in order to assure numerical stability. It should be stressed that the model is derived for lake systems having slow water level fluctuations and should not be used for tidal flows. In the applications of this study the »large water body» is the Bothnian Bay, whose sea level fluctuations are due to meteorological conditions, especially changes of the atmospheric pressure.

3. *The channel as a point loss of energy*

Friction losses and water depth along a channel between two lakes are accurately determined by gradually varied flow computations. This method is used to theoretically test whether discharge between two lakes can be determined from only lake levels and a point channel resistance factor. The steady-state condition problem of a downstream lake (known to be the downstream lake) influencing an upstream lake is known as the two-lake problem and is treated in general text books on open channel flow, *e.g.* Henderson (1966).

As an example consider a narrow shallow channel of length 100 m, bottom width 5 m, side slopes 1:10 and bank slopes 1:50. Manning's n for the channel is 0.05 and for the banks 0.1. The bottom is assumed to be flat. The depth from bottom to bank level is only 0.5 m. The discharge between the two lakes was computed for different average depth in the channel. For each fixed average depth (mean of up- and downstream levels) computations were performed for different lake level differences (energy gradient). In the step calculations along the channel an interval of 0.02 m was used. The results of the computations are shown in Table 1.

The resistance factor for the channel was computed to decrease with increasing depth. It does not, except for very shallow water depth, depend on the difference in water level between the two lakes. Thus, it is an almost unique function of the average depth in the channel. Even for a depth as low as 0.30 m and bottom

Table 1. Theoretically computed resistance factor for a channel 100 m long, bottom width 5 m, side slope 1:10, bank level 0.5 m, bank slope 1:50, Manning's n of main channel 0.05, of banks 0.1, flat bottom.

average depth (m)	lake level diff (m)	Q (m^3/s)	f $s/m^{2.5}$
1.54	0.49	20.00	0.035
1.54	0.16	11.43	0.035
0.975	0.35	7.00	0.085
0.975	0.15	4.49	0.086
0.490	0.18	1.00	0.424
0.490	0.06	0.56	0.435
0.300	0.20	0.41	1.09
0.300	0.10	0.28	1.14

width as narrow as 5 m the resistance factor is reduced by less than 5 %, when the friction slope is doubled.

If the channel is not uniform but for example much narrower close to one lake than close to the other, most of the friction losses will be restricted to the narrow part. Computations for such a channel have been carried out. A channel was assumed to be 10 m wide over 50 m and 5 m wide over the next 50 m. The cross section was rectangular and Manning's n was chosen as 0.1. The results of the computations are summarized in Table 2. The results were not affected by the direction of the flow.

Also from Table 2 it is seen that the resistance factor is an almost unique function of the average depth in the channel, computed as the mean of the depth at the inlet to and the outlet from a channel. For very low depth the resistance

Table 2. Theoretically computed resistance factor for a rectangular channel, 100 m long, width 5–10 m, Manning's n = 0.1.

average depth (m)	lake level diff (m)	Q (m^3/sec)	f ($sec/m^{2.5}$)
1.265	0.53	5.00	0.14
1.265	0.21	3.18	0.14
1.265	0.13	2.50	0.14
0.288	0.32	0.63	0.89
0.228	0.18	0.43	0.97

factor may vary within 10 % for the same depth but different friction slopes.

In the preceeding examples there was no channel bottom slope. The average depth was also the average value of the two lake levels above a datum. If the bed is sloping the difference between the average value of the lake levels and the average value of the upper and lower lake depths is a constant. Therefore, for a sloping bed the resistance factor can be related to the average value of the lake levels.

The proposed model is for flow in either direction in a channel interconnecting two lakes. The channel bed slope should be very mild and the channel relatively short. From the test examples given above it is shown that for such conditions the discharge between two lakes can be determined by a point head loss expression and that the proportionality factor (the resistance factor) is for each channel a function of the average value of the up- and downstream lake levels.

4. *Determining the resistance factor*

From measurements of lake levels and channel discharge the resistance factor, f, can be determined from eq. (4). Since these measurements can be made only for a few lake levels, the resistance factor for intermediate depth must be deter-

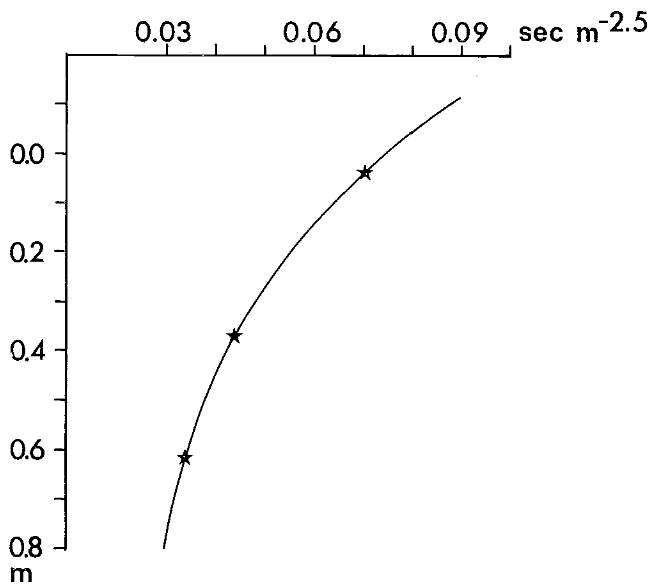

Fig. 1. An example of a f-function (resistance factor versus depth). Measured points are marked with x.

mined from interpolation. Non-uniform flow computations using Manning's formula is used to perform the interpolation. Cross sections and length of channel must be known. An example of constructing a resistance factor function is shown for a 200 m long channel between two lakes in the city of Luleå in northern Sweden. The discharge was measured at three occasions. When the mean lake level of the up- and downstream lakes was 0.62 m above mean sea level, the resistance factor f was from eq. (4) determined to 0.034 sec/m$^{2.5}$. When the mean lake level was 0.37 m f was found to be 0.044, and for the low level 0.04 f was 0.070. The bottom of the channel is at about −1.3 m. From the three measured values a resistance factor as a function of water level was constructed as shown in Fig. 1. For water levels above 0.62 m and below 0.04 m extrapolation was made using Manning's equation.

5. *Numerical test of the model*

To assure stability and accuracy of the numerical expicit scheme a short time step has to be used. To test how short a time step that is required and how accurately the model performs, results obtained from model computations using different time steps were compared. Comparison was also made with some analytical solutions as shown below.

Two identical interconnected lakes, which both are connected to the sea were considered. The system is sketched in Fig. 2. The friction factor was, for all three channels, assumed to be a function of depth, proportional to depth $^{-0.5}$, the

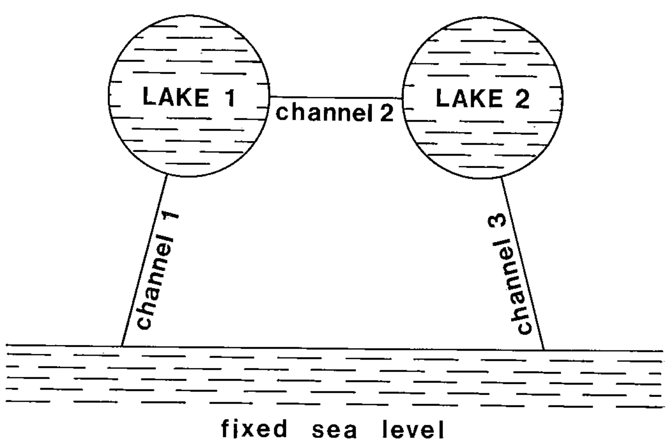

Fig. 2. A hypothetical lake system consisting of two identical lakes and three identical channels.

channel bottoms being at sea level having no slope. The lakes were assumed to have vertical shores. Initially both lake levels were at 0.20 m above sea level, while the sea level was kept constant at 0.00 m. As before, it was assumed no difference in density between the water of the lakes and of the sea. The equation describing the lake levels is

$$\frac{dh}{dt} = h^{0.5}/Af = h/(c \cdot A \cdot 2^{0.5}) \tag{7}$$

where h = lake level, A = lake area and c = a proportionality constant. The analytical solution is

$$h = h_0\, e^{-t/(c \cdot A \cdot 2^{0.5})} \tag{8}$$

where h_0 = initial lake level.

For a lake area of 1.3 km^2 and c = 0.2 s/m^2 in the f-function the results of the numerical explicit computations were found to follow the analytical solution even for a time step of 6 hours.

For more complex lake systems it is not possible to find analytical solutions for the lake levels. Instead lake levels computed by the explicit model for different time steps were compared. For a system of six lakes with new input data every day it was found that a time step of 2 hours was sufficient to assure an accuracy better than 0.01 m when computing lake levels. In this test example the »boundary» level, *i.e.* the known level of a large water body, usually the sea, was allowed to change 0.50 m over a day.

6. *Application of the model*

The proposed model was tested on a lake system in Luleå. There are six lakes in the lake system. The lake system is sketched in Fig. 3. The lake system is connected to the sea, *i.e.* the Bothnian Bay, having almost fresh water, through a wide sound in the north and through a very narrow and shallow canal in the south. The lake level of the large water body, Lake Brändö, in the north closely follows the sea level. The southern part of Lake Brändö constitutes a lake of itself called Lake Sörfjärd. Lake Björsby has been observed to be the uppermost lake, but all the lake levels can be below the level of the sea, when high sea water levels are recorded in the Bothnian Bay. The main external inflow to the southern lakes is the River Holmsundet. The external inflow to Lake Björkskata is from the urban drainage system of some parts of the city of Luleå. During snowmelt, there is also some external flow to Lake Sinkfjärd and Lake Sörfjärd. Some

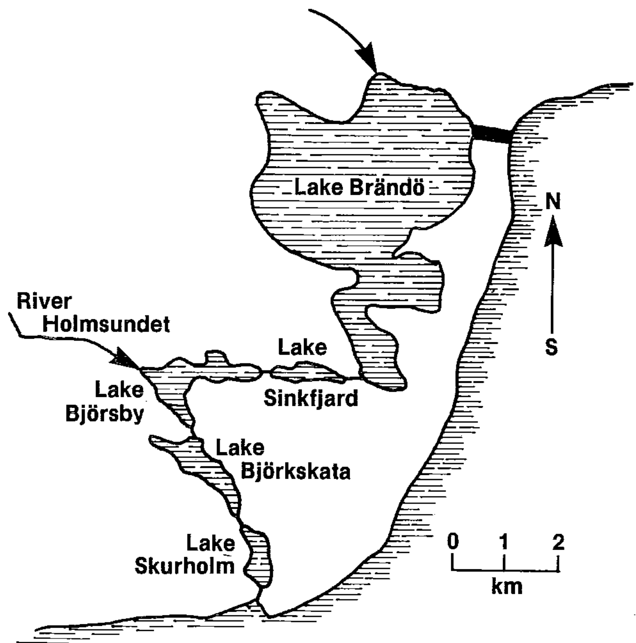

Fig. 3. The lake system under consideration, the city lakes of Luleå, Sweden.

applications of the model to this lake system are described by BENGTSSON (1984).

Cross sections over some parts of the lakes, cross sections of the channels as well as leveling data of the bottom profile along the channels were available. The discharge in all the channels between the lakes were measured on at least two occasions, and from simultaneous measurements of lake levels the resistance factor was calculated using eq. (4). Manning's formula was used for interpolation and extrapolation when estimating f for other lake levels. Friction factors in all the channels in the system as a function of depth is given in a technical report, BENGTSSON (1980).

As seen from Fig. 3 the lake system is interconnected with the sea. During dry periods and when there are rather fast sea level fluctuations, water from the Bay of Bothnia is pressed into the lake system. During snowmelt, when there is large external inflow into the lake system from upstream catchments, the lake levels are almost independent of the sea level. The model was calibrated for snowmelt conditions. For this situation all lake levels were computed very accurately.

The model was tested for an independent period with almost steady-state lake

Table 3. Comparison between measured and computed conditions in the Lule City Lake System, May 17–19, 1981.

	water level (m)		flow (m^3/s)	
	measured	computed	measured	computed
canal sea – Lake Skurholm	0.05	0.00	7.2	6.7
canal Lake Skurholm – Lake Björkskata	0.37	0.38	6.2	6.2
canal Lake Björkskata – Lake Björsby	–	0.48	4.5	4.7
canal Lake Björsby – Lake Sinkfjärd	0.50	0.52	1.8	1.6
canal Lake Sinkfjärd – Lake Brändö	−0.23	−0.21	3.0	2.8

levels. The period was mid-May 1981, when the snow had disappeared from open areas and snowmelt continued only in the forested areas. Observations of sea level, inflow to Lake Björsby from the River Holmsundet and the flow in a creek entering Lake Sörfjärd were available. The small external inflow to Lake Sinksund was related to the flow in the creek entering Lake Sörfjärd. Since it was difficult to estimate the input of urban drainage water to the two lower lakes, Lake Björkskata and Lake Skurholm, the simulations were not started until the urban areas were snowfree, May 11. The baseflow into these two lakes was estimated to be 1.5 and 0.5 m^3/sec, respectively. The computed and measured almost steady state conditions, which prevailed May 17–19 are compared in Table 3. Lake Björsby remained the uppermost lake at a constant level through the whole period. The water level of Lake Brändö, which is almost a part of the Luleå archipelago, could, however, not be considered to be steady-state.

A test of the performance of the model over a longer period was made by a continuous simulation for the period Oct. 10–Nov. 9, 1978. The period was dry. The daily sea levels were known. The only important river inflow, *i.e.* through the River Holmsundet, was estimated to be constant and equal to 2.0 m^3/sec, which was found from weekly water stage measurements carried out 1.5 km upstreams Lake Björsby. A comparison between observed and computed lake levels is made in Fig. 4. The agreement seems to be good. However, lake level data existed only for five occasions during the period. No data was available for the days, when the sea level dropped to or rose to rather extreme values.

A second test was made on data from a period of 19 days in September 1983. Daily lake level data and discharge of the river Holmsundet were available. A com-

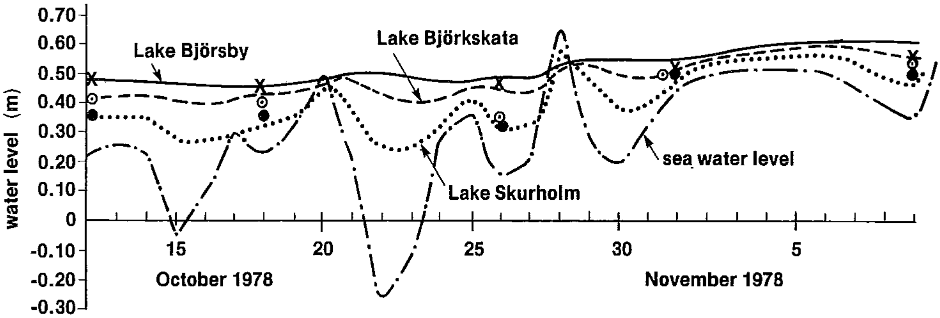

Fig. 4. Observed (X for Lake Björsby, circle with dot for Lake Björkskata, filled circle for Lake Skurholm) and computed lake levels Oct./Nov. 1978.

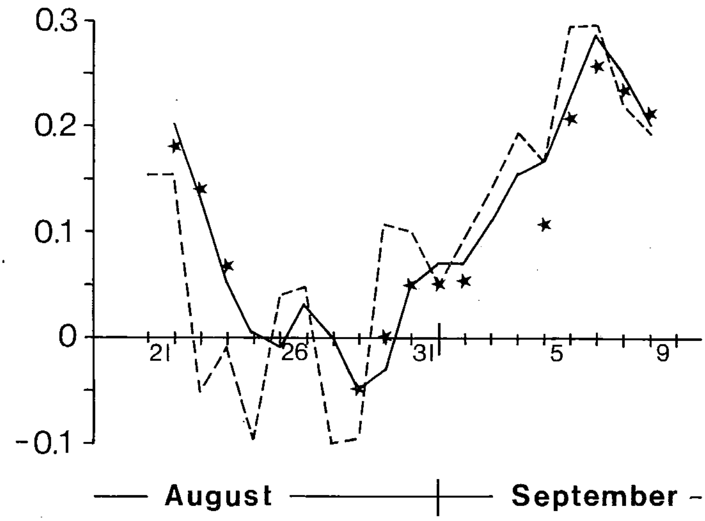

Fig. 5. Observed (X) and computed (solid line) daily water level of Lake Björsby, and observed sea level (broken line), Aug./Sept. 1983.

parison between measured and computed levels of Lake Björsbyn is made in Fig. 5. Lake level data from the lake closest to the sea, Lake Skurholm, was not available. The agreement between observed and computed lake levels is good except for September 5. Part of the explanation of the agreement is that the lake system is a damped system. This is discussed further in the next section.

7. *Regression of lake levels*

The water levels of the three uppermost lakes in the Luleå city lake system, *i.e.* the lakes which usually have the highest lake levels, follow each other rather closely. The lakes below these lakes, one in each direction, are so small that their storage capability is negligable. The level of the very large northern lake, Lake Brändö, closely follows the sea level. Therefore, the lake level of the uppermost lake, Lake Björsby, can be described as a function of previous lake levels of that lake, external inflow from the upstream catchment and the sea level. Linear regression results in the regression model

$$h^{(n+1)} = r_t h^{(n)} + r_s h_s^{(n)} + r_q Q^{(n)} \tag{9}$$

where r_t = autoregression coefficient, r_s = regression coefficient for parameter sea level and r_q = regression coefficient for parameter river inflow. Upper indexes within parenthesis refer to time step. Normalized variables are used.

For a time lag of one day the autoregression coefficient, r_t, was found to be 0.80. The regression coefficient for sea level, r_s, was 0.41. Since only few data was available on discharge in the main river, the regression coefficient r_q could not be given a value. The autocorrelation and the correlation with the sea level explain 80 % of the variance of the lake level of the uppermost lake. The high autocorrelation shows that the system is damped. Therefore, the deterministic model should be expected to give accurate levels of the uppermost lake, even if the friction factors are not estimated correctly.

8. *Conclusions*

A system of interconnected lakes which are also connected to a large water body of known water level is mathematically treated by a continuity equation for each lake and a point loss energy equation for each interconnecting channel. The discharge between two lakes is taken as proportional to the square root of the difference in lake level between the lakes. The proportionality factor for a channel is except for extremely shallow depth a unique function of the water depth.

The proposed mathematical model is suitable for determining lake level fluctuations as a function of external inflows and sea level fluctuations. Comparisons between computed and measured lake levels show that the model should be suitable for solving engineering problems.

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NOTATION

The following symbols are used in this paper:

A	lake surface area, channel cross section area
B	channel width
f	»resistance factor» (sec/m$^{2.5}$)
h	lake level, water depth
h_0	initial lake level
dh	difference in potential energy (lake level) between two lakes
i	lake number index
L	channel length
n	Manning's n
(n)	time level
Q	channel discharge
Q_{ei}	external inflow to lake no. i
$Q_{i-1,i}$	flow from lake no. (i–1) to no. i
R	hydraulic radius
r_t	autoregression coefficient
r_s	regression coefficient for parameter sea level
r_q	regression coefficient for parameter river discharge
S	lake storage
S_f	friction slope
t	time coordinate
x	distance coordinate