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ALGORITHMS FOR RAYLEIGH AND LOVE WAVE DISPERSION COMPUTATIONS

by

PEKKA SAASTAMOINEN

Department of Geophysics
University of Helsinki

and

MATTI TARVAINEN

Institute of Seismology
University of Helsinki

Abstract

Operator spectral decompositions of Rayleigh and Love wave propagators have been applied to formulate algorithms of implicit function type to compute the dispersion related quantities, phase and group velocities, as well as their partial derivatives with respect to the structural parameters of the plane layered waveguide which models the crust and upper mantle of the Earth.

1. *Introduction*

In the inversion of surface wave observations (phase and group velocities), it is essential to have reliable algorithms to solve the corresponding direct problems, namely theoretical phase and/or group velocities and their partial derivatives with respect to the structural parameters of the waveguide.

At present there are many accurate and comparatively fast algorithms to compute phase velocities (see ABO-ZENA (1979) and KENNETT and CLARKE (1983)). For the computation of group velocities and the relevant partial derivatives with respect to structural parameters it is possible to choose any of the following methods:

1. numerical differentiation;
2. variational and perturbational techniques;
3. implicit function techniques.

Each of these methods, with their merits and shortcomings, has been described briefly by NOVOTNY (1976).

In this paper we present algorithms of implicit function type to compute the dispersion related quantities, phase velocities and group velocities, and their partial derivatives with respect to structural parameters of the waveguide. All the algorithms are based on the following:

1. the computation of the Rayleigh wave dispersion function and its derivatives relies on the use of propagators of Abo-Zena type;
2. in both the Rayleigh and the Love cases the vertical propagators are used in their operator spectral representations.

2. *Statement of the dispersion related boundary value problems*

The geometry of the underlying waveguide C is presented schematically in fig. 1. The isotropic, elastic and homogeneous waveguide consists of the layers, C_j, j = 1,2,...,n, and the halfspace C_{n+1}. The layers are separated by the interfaces Γ_j, j = 1,2,...,n. The waveguide C is bounded from above by the free surface Γ_0.

We turn next to the proper formulation of the R- and L-wave boundary value problems. The considerations presented below are valid for an arbitrary cross-section $S^{\perp}$ of the waveguide. Firstly, the equation

$$\frac{d\overset{R,L}{s}}{dz} = \overset{R,L}{A}\ \overset{R,L}{s} \tag{2.1}$$

allows for the propagation of spectral information between any two points of the cross-section. Secondly, the state vectors take on the form

$$\overset{R}{s} = \begin{pmatrix} v_z \\ v_R \\ t_z \\ t_R \end{pmatrix} \quad \text{or} \quad \overset{L}{s} = \begin{pmatrix} v_T \\ t_T \end{pmatrix} \tag{2.2}$$

depending upon whether we are considering R- or L-waves. In (2.2), v_z, v_R and v_T are the vertical, radial and transversal components of the velocity spectral vector. t_z, t_R and t_T are, in turn, the corresponding components of the stress

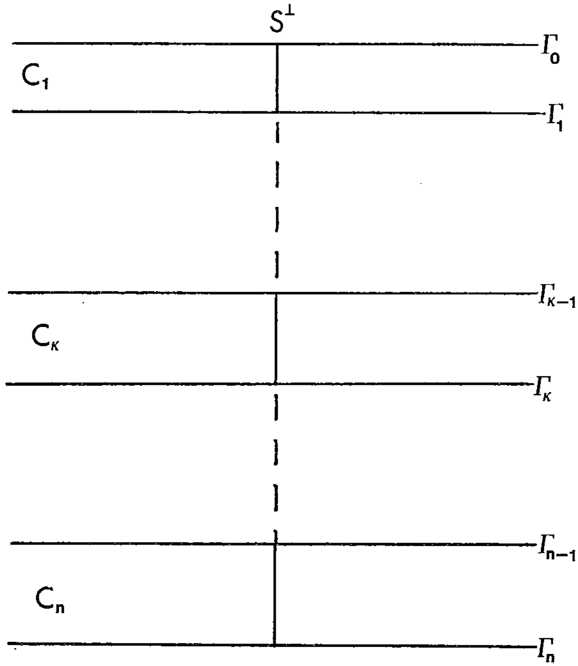

Fig. 1. Geometry of the waveguide.

spectral vector. Similarly, the matrix $\overset{R,L}{A}$ has two meanings, namely

$$\overset{R}{A} = \begin{pmatrix} 0 & \frac{1}{c}\left(1-2\frac{\beta^2}{\alpha^2}\right) & \frac{1}{\rho\alpha^2} & 0 \\ \frac{1}{c} & 0 & 0 & \frac{1}{\rho\beta^2} \\ \rho & 0 & 0 & \frac{1}{c} \\ 0 & \rho\left(1-\frac{4\beta^2}{c^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\right) & \frac{1}{c}\left(1-\frac{2\beta^2}{\alpha^2}\right) & 0 \end{pmatrix} \text{ and } \overset{L}{A} = \begin{pmatrix} 0 & \frac{1}{\rho\beta^2} \\ \rho\left(1-\frac{\beta^2}{c^2}\right) & 0 \end{pmatrix} \quad (2.3)$$

in the R- and L-cases, respectively. In (2.3), C is the phase velocity of the surface wave in question, α is P wave velocity, β is S wave velocity and ρ is density. Due to the special geometry of $S^{\perp}$ the matrices $\overset{R}{A}$ and $\overset{L}{A}$ should be regarded as layer-wise constant.

We still lack the appropriate boundary conditions at Γ_j, $j = 0, 1, 2, ..., n$. These are stated as follows:

1. Vanishing of the stress spectral vector at the free surface Γ_0, *i.e.*,

$$\underset{1}{\overset{R}{s}}|_{\Gamma_0} = \begin{pmatrix} \underset{1}{\check{z}}|_{\Gamma_0} \\ \underset{1}{\check{\bar{R}}}|_{\Gamma_0} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \underset{1}{\overset{L}{s}}|_{\Gamma_0} = \begin{pmatrix} \underset{1}{\check{T}}|_{\Gamma_0} \\ 0 \end{pmatrix} \tag{2.4}$$

2. Continuity of the state vectors at the interfaces Γ_j, *i.e.*,

$$\underset{j}{\overset{R}{s}}|_{\Gamma_j} = \underset{j+1}{\overset{R}{s}}|_{\Gamma_j} \quad \text{and} \quad \underset{j}{\overset{L}{s}}|_{\Gamma_j} = \underset{j+1}{\overset{L}{s}}|_{\Gamma_j} \tag{2.5}$$

for $j = 1, 2, ..., n$. It should be noted that the notation $s_{|\Gamma}$, in the expressions (2.4) and (2.5), means restriction of s to the manifold Γ. Further, the subscripts refer to the various layers of the problem.

3. As a further boundary condition, only waves propagating outward are allowed in the halfspace below the waveguide.

3. *Layer propagators and their operator spectral representations*

The layer propagators related to our differential equation (2.1) are solutions of the following initial value problems

$$\frac{d\overset{R}{P}}{dz} = \overset{R}{A}\overset{R}{P}; \quad \overset{R}{P}(z_0) = \overset{R}{I} \quad \text{and} \quad \frac{d\overset{L}{P}}{dz} = \overset{L}{A}\overset{L}{P}; \quad \overset{L}{P}(z_0) = \overset{L}{I}$$

where $\overset{R}{I}$ and $\overset{L}{I}$ are the relevant unit matrices. Layer propagators were introduced by GILBERT and BACKUS (1966). They form an effective way to handle dispersion related boundary value problems in 1D situations. In the case of R- and L-wave boundary value problems, the layer propagators are of the form

$$\underset{j}{\overset{R}{P}} = e^{i\frac{2\pi}{T}\underset{j}{\overset{R}{A}}\underset{j}{d}} \quad \text{and} \quad \underset{j}{\overset{L}{P}} = e^{i\frac{2\pi}{T}\underset{j}{\overset{L}{A}}\underset{j}{d}} \tag{3.1}$$

for $j = 1, 2, ..., n$. In (3.1), T is the period of oscillation and $\underset{j}{d}$ is the thickness of the jth layer.

In addition to introducing the layer propagators, the purpose of the present section is to develop operator spectral representations of these propagators. According to KATO (1966) an operator function $f(A)$ has a representation as the operator integral

$$f(A) = \frac{1}{2\pi i} \int_{\sigma_A} f(\nu)\,(\nu - A)^{-1}\, d\nu \tag{3.2}$$

along the complex contour σ_A around the spectrum of A. If the spectrum consists of simple eigenvalues, then (3.2) will take the form,

$$f(A) = \sum_m f(\nu_m)\Pi_m \tag{3.3}$$

as the residue sum over the eigenvalues of A. In addition, the eigenprojectors Π_m are given as the dyadic composition

$$\Pi_m = e_m e^m \tag{3.4}$$

where e_m, $m = 1,2,\ldots$, are the eigenvectors of A and e^m, $m = 1,2,\ldots$ are the eigenvectors of the adjoint operator A^T.

Under (3.3), our layer propagators have the eigen representations

$$\overset{R}{\underset{j}{P}} = \sum_{m=1}^{4} e^{i\frac{2\pi}{T}\overset{R}{\underset{j}{\nu}}_m \underset{j}{d}}\, \overset{R}{\underset{j}{\Pi}}_m \qquad \text{and} \qquad \overset{L}{\underset{j}{P}} = \sum_{m=1}^{2} e^{i\frac{2\pi}{T}\overset{L}{\underset{j}{\nu}}_n \underset{j}{d}}\, \overset{L}{\underset{j}{\Pi}}_m \tag{3.5}$$

for $j = 1,2,\ldots,n$. The actual representation of the eigenvalues as well as the eigenvectors are given in Appendix A.

4. *Rayleigh and Love wave dispersion functions*

The properties of the propagators together with the boundary conditions (2.5) makes it possible to express the relationship between $\underset{n}{s}|\Gamma_n$ and $\underset{1}{s}|\Gamma_0$ as follows

$$\overset{R}{\underset{n}{s}}|\Gamma_n = \overset{R}{P} \cdot \overset{R}{\underset{1}{s}}|\Gamma_0 \qquad \text{and} \qquad \overset{L}{\underset{n}{s}}|\Gamma_n = \overset{L}{P} \cdot \overset{L}{\underset{1}{s}}|\Gamma_0 \tag{4.1}$$

In (4.1),

$$\overset{R}{P} = \overset{R}{P}_n \cdot \overset{R}{P}_{n-1} \cdots \overset{R}{P}_1 \qquad \text{and} \qquad \overset{L}{P} = \overset{L}{P}_n \cdot \overset{L}{P}_{n-1} \cdots \overset{L}{P}_1 \tag{4.2}$$

In (4.1) and (4.2) the use of the »dot» operation follows the usual lines of Gibbsian dyadic calculus.

We already know that (4.1) and (4.2) satisfy the differential equations and the boundary conditions (2.5). For complete fulfilment of the boundary conditions stated in section 2 we are still left with (2.4) as well as the outward propagation condition. Due to the boundary condition (2.4) we find that

$$\underset{1}{\overset{R}{s}}|_{\Gamma_0} = \overset{R\prime}{e}\overset{R\prime}{e} \cdot \underset{1}{\overset{R}{s}}|_{\Gamma_0} + \overset{R\prime\prime}{e}\overset{R\prime\prime}{e} \cdot \underset{1}{\overset{R}{s}}|_{\Gamma_0} \quad \text{and} \quad \underset{1}{\overset{L}{s}}|_{\Gamma_0} = \overset{L\prime}{e}\overset{L\prime}{e} \cdot \underset{1}{\overset{L}{s}}|_{\Gamma_0} \tag{4.3}$$

where

$$\overset{R\prime}{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \overset{R\prime\prime}{e} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \overset{L\prime}{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.4}$$

are Cartesian vectors. The outward propagation condition is seen to be equivalent to the cancellation of all upward propagating waves in the halfspace. Consequently,

$$\underset{n+1}{\overset{R_1}{e}} \cdot \underset{n}{\overset{R}{s}}|_{\Gamma_n} = 0 \quad \text{and} \quad \underset{n+1}{\overset{R_2}{e}} \cdot \underset{n}{\overset{R}{s}}|_{\Gamma_n} = 0 \tag{4.5}$$

for R-waves, and

$$\underset{n+1}{\overset{L_1}{e}} \cdot \underset{n}{\overset{L}{s}}|_{\Gamma_n} = 0 \tag{4.6}$$

for L-waves.

The conditions (4.1) – (4.6) are equivalent to the original boundary value problem which was stated in section 2. Combining these expressions we find the eigenvalue problems

$$\begin{pmatrix} \underset{n+1}{\overset{R_1}{e}} \cdot \overset{R}{P} \cdot \overset{R\prime}{e} & \underset{n+1}{\overset{R_1}{e}} \cdot \overset{R}{P} \cdot \overset{R\prime\prime}{e} \\ \underset{n+1}{\overset{R_2}{e}} \cdot \overset{R}{P} \cdot \overset{R\prime}{e} & \underset{n+1}{\overset{R_2}{e}} \cdot \overset{R}{P} \cdot \overset{R\prime\prime}{e} \end{pmatrix} \begin{pmatrix} \overset{R\prime}{e} \cdot \underset{1}{\overset{R}{s}}|_{\Gamma_0} \\ \overset{R\prime\prime}{e} \cdot \underset{1}{\overset{R}{s}}|_{\Gamma_0} \end{pmatrix} = 0 \tag{4.7}$$

for R-waves and

$$\left(\underset{n+1}{\overset{L_1}{e}} \cdot \overset{L}{P} \cdot \overset{L}{e}'\right)\left(\overset{L}{e}' \cdot \underset{1}{\overset{L}{s}}\Big|_{\Gamma_0}\right) = 0 \tag{4.8}$$

for L-waves. The solutions to the above problems depend non-linearly on the eigenvalues c and T and linearly on the eigenvectors. The solvability of (4.7) and (4.8) with respect to the eigenvectors leads to the dispersion equations

$$\overset{R}{F}(c, T; m) \equiv det \begin{pmatrix} \underset{n+1}{\overset{R_1}{e}} \cdot \overset{R}{P} \cdot \overset{R}{e}' & \underset{n+1}{\overset{R_1}{e}} \cdot \overset{R}{P} \cdot \overset{R}{e}'' \\ \underset{n+1}{\overset{R_2}{e}} \cdot \overset{R}{P} \cdot \overset{R}{e}' & \underset{n+1}{\overset{R_2}{e}} \cdot \overset{R}{P} \cdot \overset{R}{e}'' \end{pmatrix} = 0 \tag{4.9}$$

for R-waves and

$$\overset{L}{F}(c, T; m) \equiv \underset{n+1}{\overset{L_1}{e}} \cdot \overset{L}{P} \cdot \overset{L}{e}' = 0 \tag{4.10}$$

for L-waves. The dependence on the structural parameters has been expressed through the vector

$$m = (\alpha_1 \beta_1 \rho_1 d_1 \ldots \alpha_{n+1} \beta_{n+1} \rho_{n+1})^T \tag{4.11}$$

We next develop computational algorithms for evaluation of the dispersion functions in (4.9) and (4.10). Starting with R-waves, consideration of (4.9) results in

$$\overset{R}{F}(c, T; m) = \overset{R}{e}' \cdot \overset{R}{\mathbb{P}} \cdot \underset{n+1}{\overset{R}{Y}} \cdot \overset{R}{e}'' \tag{4.12}$$

In (4.12), we have introduced the new operator

$$\overset{R}{\mathbb{P}} \cdot \underset{n+1}{\overset{R}{Y}} = \overset{R}{P}^T \cdot \underset{n+1}{\overset{R}{Y}} \cdot \overset{R}{P} \tag{4.13}$$

called here the Abo-Zena propagator. $\underset{n+1}{\overset{R}{Y}}$ means the dyadic expression

$$\underset{n+1}{\overset{R}{Y}} = \underset{n+1}{\overset{R_1}{e}}\,\underset{n+1}{\overset{R_2}{e}} - \underset{n+1}{\overset{R_2}{e}}\,\underset{n+1}{\overset{R_1}{e}} \tag{4.14}$$

In the L-wave case corresponding considerations lead to the dispersion function

$$\overset{L}{F}(c, T; m) = \overset{L}{e}' \cdot \overset{L}{\mathbb{P}} \cdot \underset{n+1}{\overset{L}{Y}}\ , \tag{4.15}$$

where

$$\overset{L}{\mathbb{P}} \cdot \underset{n+1}{\overset{L}{Y}} = \overset{L_T}{P} \cdot \underset{n+1}{\overset{L}{Y}} \tag{4.16}$$

and

$$\underset{n+1}{\overset{L}{Y}} = \underset{n+1}{\overset{L_1}{e}} \tag{4.17}$$

The actual computation of the dispersion functions (4.12) and (4.15) may conveniently be carried out iteratively, as follows

$$\underset{j}{\overset{R}{Y}} = \underset{j}{\overset{R}{\mathbb{P}}} \cdot \underset{j+1}{\overset{R}{Y}} \quad \text{and} \quad \underset{j}{\overset{L}{Y}} = \underset{j}{\overset{L}{\mathbb{P}}} \cdot \underset{j+1}{\overset{L}{Y}} \tag{4.18}$$

for $j = n, n-1, \ldots, 1$. The initial values of the sequences are those given in (4.14) and (4.17), respectively. After the iterations have been carried out, the dispersion functions are obtained from

$$\overset{R}{F}(c, T; m) = \overset{R}{e}' \cdot \underset{1}{\overset{R}{Y}} \cdot \overset{R}{e}'' \quad \text{and} \quad \overset{L}{F}(c, T; m) = \overset{L}{e}' \cdot \underset{1}{\overset{L}{Y}} \tag{4.19}$$

The operator spectral representations of the propagators $\underset{j}{\overset{R}{\mathbb{P}}}$ and $\underset{j}{\overset{L}{\mathbb{P}}}$ are presented in Appendix B.

5. *T-derivatives of the dispersion functions*

We derive now computational algorithms for the evalutions of the partial derivatives of the dispersion functions with respect to the period T. This task has been accomplished through the differentiation of (4.18) and (4.19) with respect to T, which results in the iterative sequences

$$\underset{j}{\overset{R}{Y}}_{,T} = \underset{j}{\overset{R}{\mathbb{P}}}_{,T} \cdot \underset{j+1}{\overset{R}{Y}} + \underset{j}{\overset{R}{\mathbb{P}}} \cdot \underset{j+1}{\overset{R}{Y}}_{,T} \quad \text{and} \quad \underset{j}{\overset{L}{Y}}_{,T} = \underset{j}{\overset{L}{\mathbb{P}}}_{,T} \cdot \underset{j+1}{\overset{L}{Y}} + \underset{j}{\overset{L}{\mathbb{P}}} \cdot \underset{j+1}{\overset{L}{Y}}_{,T} \tag{5.1}$$

for $j = n, n-1, \ldots, 1$ and the evaluations

$$\overset{R}{F}_{,T}(c, T; m) = \overset{R}{e}' \cdot \underset{1}{\overset{R}{Y}}_{,T} \cdot \overset{R}{e}'' \quad \text{and} \quad \overset{L}{F}_{,T}(c, T; m) = \overset{L}{e}' \cdot \underset{1}{\overset{L}{Y}}_{,T} \cdot \overset{L}{e}'' \tag{5.2}$$

The initial values of the iteration sequences are those given in (4.14) and (4.17). The new propagators, $\overset{R}{\mathbb{P}}_{,T}$ and $\overset{L}{\mathbb{P}}_{,T}$, are presented in Appendix B.

6. *c-derivatives of the dispersion functions*

In deriving computational algorithms for the c-derivatives of the dispersion functions it is more straightforward to differentiate the expressions (4.18) and (4.19). Carrying out the c-differentiations results in the iterative sequences

$$\underset{j}{\overset{R}{Y}}_{,c} = \underset{j}{\overset{R}{\mathbb{P}}}_{,c} \cdot \underset{j+1}{\overset{R}{Y}} + \underset{j}{\overset{R}{\mathbb{P}}} \cdot \underset{j+1}{\overset{R}{Y}}_{,c} \quad \text{and} \quad \underset{j}{\overset{L}{Y}}_{,c} = \underset{j}{\overset{L}{\mathbb{P}}}_{,c} \cdot \underset{j+1}{\overset{L}{Y}} + \underset{j}{\overset{L}{\mathbb{P}}} \cdot \underset{j+1}{\overset{L}{Y}}_{,c} \tag{6.1}$$

for $j = n, n-1, \ldots, 1$, and the evaluations

$$\overset{R}{F}_{,c}(c, T; m) = \overset{R}{e}{}' \cdot \underset{1}{\overset{R}{Y}}_{,c} \cdot \overset{R}{e}{}'' \quad \text{and} \quad \overset{L}{F}_{,c}(c, T; m) = \overset{L}{e}{}' \cdot \underset{1}{\overset{L}{Y}}_{,c} \tag{6.2}$$

The initial values for (6.1) are found to be

$$\underset{n+1}{\overset{R}{Y}}_{,c} = \underset{n+1}{\overset{R_1}{e}}_{,c} \; \underset{n+1}{\overset{R_2}{e}} + \underset{n+1}{\overset{R_1}{e}} \; \underset{n+2}{\overset{R_2}{e}}_{,c} - \underset{n+1}{\overset{R_2}{e}}_{,c} \; \underset{n+1}{\overset{R_1}{e}} - \underset{n+1}{\overset{R_2}{e}} \; \underset{n+1}{\overset{R_1}{e}}_{,c} \tag{6.3}$$

and

$$\underset{n+1}{\overset{L}{Y}}_{,c} = \underset{n+1}{\overset{L_1}{e}}_{,c} \tag{6.4}$$

In (6.3) and (6.4) we need explicit expressions for the c-derivatives of the eigenvectors. These are found in Appendix A. In addition, the operator spectral expressions of the *c*-propagators $\overset{R}{\mathbb{P}}_{,c}$ and $\overset{L}{\mathbb{P}}_{,c}$ needed in (6.1) are to be found in Appendix B.

7. *Partial derivatives of the dispersion functions with respect to structural parameters*

The vectorial expression (4.11) for the structural parameters implies the obvious correspondence

$$m_{4j-l} = \begin{cases} \alpha_j & , \quad l = 3 \\ \beta_j & , \quad l = 2 \\ \rho_j & , \quad l = 1 \\ d_j & , \quad l = 0 \end{cases}$$

This simple relationship is extremely useful in understanding the subsequent partial derivatives expressed in terms of the component m_{4j-l}.

A treatment of (4.18) similar to the one carried out in the previous sections gives us the following types of iterative sequences

$$1. \quad \overset{R}{\underset{1}{Y}}_{,m_{4j-l}} = \overset{R}{\underset{1}{\mathbb{P}}} \cdot \overset{R}{\underset{2}{Y}}_{,m_{4j-l}} \quad \text{and} \quad \overset{L}{\underset{1}{Y}}_{,m_{4j-l}} = \overset{L}{\underset{1}{\mathbb{P}}} \cdot \overset{L}{\underset{2}{Y}}_{,m_{4j-l}} \tag{7.1}$$

$$\vdots \qquad\qquad \vdots$$

$$\overset{R}{\underset{j}{Y}}_{,m_{4j-l}} = \overset{R}{\underset{j}{\mathbb{P}}}_{,m_{4j-l}} \cdot \overset{R}{\underset{j+1}{Y}} \qquad\qquad \overset{L}{\underset{j}{Y}}_{,m_{4j-l}} = \overset{L}{\underset{j}{\mathbb{P}}}_{,m_{4j-l}} \cdot \overset{L}{\underset{j+1}{Y}}$$

$$\vdots \qquad\qquad \vdots$$

$$\overset{R}{\underset{n}{Y}} = \overset{R}{\underset{n}{\mathbb{P}}} \cdot \overset{R}{\underset{n+1}{Y}} \qquad\qquad \overset{L}{\underset{n}{Y}} = \overset{L}{\underset{n}{\mathbb{P}}} \cdot \overset{L}{\underset{n+1}{Y}}$$

for $1 \leqq j \leqq n$ and $l = 0,1,2,3$.

$$2. \quad \overset{R}{\underset{j}{Y}}_{,m_{4(n+1)-l}} = \overset{R}{\underset{j}{\mathbb{P}}} \cdot \overset{R}{\underset{j+1}{Y}}_{,m_{4(n+1)-l}} \quad \text{and} \quad \overset{L}{\underset{j}{Y}}_{,m_{4(n+1)-l}} = \overset{L}{\underset{1}{\mathbb{P}}} \cdot \overset{L}{\underset{j+1}{Y}}_{,m_{4(n+1)-l}} \tag{7.2}$$

for $j = n, n-1, ..., 1$ and $l = 1,2,3$. The derivatives of (4.19) result in the expressions

$$\overset{R}{F}_{,m_{4j-l}}(c, T; m) = \overset{R}{e}' \cdot \overset{R}{\underset{1}{Y}}_{,m_{4j-l}} \cdot \overset{R}{e}'' \quad \text{and} \quad \overset{L}{F}_{,m_{4j-l}}(c, T; m) = \overset{L}{e}' \cdot \overset{L}{\underset{1}{Y}}_{,m_{4j-l}} \tag{7.3}$$

for $j = 1,2,..., n+1$ and $l = 0,1,2,3$. The initial values of the sequences (7.2) are of the form

$$\overset{R}{\underset{n+1}{Y}}_{,m_{4(n+1)-l}} = \begin{cases} \overset{R_1}{\underset{n+1}{e}}_{,\alpha_{n+1}} \overset{R_2}{\underset{n+1}{e}} + \overset{R_1}{\underset{n+1}{e}} \overset{R_2}{\underset{n+1}{e}}_{,\alpha_{n+1}} - \overset{R_2}{\underset{n+1}{e}}_{,\alpha_{n+1}} \overset{R_1}{\underset{n+1}{e}} - \overset{R_2}{\underset{n+1}{e}} \overset{R_1}{\underset{n+1}{e}}_{,\alpha_{n+1}}, & l = 3 \\ \overset{R_1}{\underset{n+1}{e}}_{,\beta_{n+1}} \overset{R_2}{\underset{n+1}{e}} + \overset{R_1}{\underset{n+1}{e}} \overset{R_2}{\underset{n+1}{e}}_{,\beta_{n+1}} - \overset{R_2}{\underset{n+1}{e}}_{,\beta_{n+1}} \overset{R_1}{\underset{n+1}{e}} - \overset{R_2}{\underset{n+1}{e}} \overset{R_1}{\underset{n+1}{e}}_{,\beta_{n+1}}, & l = 2 \\ \overset{R_1}{\underset{n+1}{e}}_{,\rho_{n+1}} \overset{R_2}{\underset{n+1}{e}} + \overset{R_1}{\underset{n+1}{e}} \overset{R_2}{\underset{n+1}{e}}_{,\rho_{n+1}} - \overset{R_2}{\underset{n+1}{e}}_{,\rho_{n+1}} \overset{R_1}{\underset{n+1}{e}} - \overset{R_2}{\underset{n+1}{e}} \overset{R_1}{\underset{n+1}{e}}_{,\rho_{n+1}}, & l = 1 \end{cases} \tag{7.4}$$

for R-waves and of the form

$$\overset{L}{Y}_{n+1},_{m_{4(n+1)-l}} = \begin{cases} \overset{L_1}{e}_{n+1},_{\beta_{n+1}} & , \quad l = 2 \\ \overset{L_1}{e}_{n+1},_{\rho_{n+1}} & , \quad l = 1 \end{cases} \tag{7.5}$$

for L-waves. The partial derivatives of the eigenvectors with respect to the structural parameters are given in Appendix A. The expressions for the structural parameter propagators, $\overset{R}{\mathbb{P}},_{m_{4j-l}}$ and $\overset{L}{\mathbb{P}},_{m_{4j-l}}$, are given in Appendix B.

8. *Phase and group velocities and their derivatives with respect to the structural parameters*

In (4.18) and (4.19) we have algorithms for evaluation of the dispersion functions. Our next task is to formulate computational algorithms to solve the dispersion equations

$$\overset{R}{F}(c, T; m) = 0 \quad \text{and} \quad \overset{L}{F}(c, T; m) = 0$$

for the phase velocity c. In this paper we have solved the problem numerically using an algorithm which, for fixed T and m, seeks an approximate value of c. With this estimate as the initial value, Newton's iteration method gives the final value to any specified accuracy. We see that our algorithms give the R- and L-wave dispersions

$$\overset{R}{c} = \overset{R}{c}(T; m) \quad \text{and} \quad \overset{L}{c} = \overset{L}{c}(T; m)$$

Having determined the phase velocity dispersion, we turn to the group velocity dispersion. It is well known that this can be accomplished with the aid od the formula

$$v_g(T; m) = \frac{cF,_c}{cF,_c - TF,_T}\, c \tag{8.1}$$

We have already at our disposal all the algorithms, determination of phase velocities and the evaluation of the c- and T-derivatives of the dispersion functions, required to evaluate (8.1). We may therefore regard also the group velocity dispersions

$$\overset{R}{\mathrm{v}}_g = \overset{R}{\mathrm{v}}_g(T; m) \quad \text{and} \quad \overset{L}{\mathrm{v}}_g = \overset{L}{\mathrm{v}}_g(T; m) \tag{8.2}$$

as solved.

Our next task is to determine the structural parameter derivatives, $D_m \overset{R}{c}$ and $D_m \overset{L}{c}$, of the phase velocities. Differentiation of the implicit functions results in the expressions

$$D_m \overset{R}{c} = -\frac{D_m \overset{R}{F}}{\overset{R}{F}_{,c}} \quad \text{and} \quad D_m \overset{L}{c} = -\frac{D_m \overset{L}{F}}{\overset{L}{F}_{,c}} \tag{8.3}$$

for the derivatives required. In (8.3),

$$D_m \overset{R}{F} = (\overset{R}{F}_{,m_1} \dots \overset{R}{F}_{,m_{4(n+1)-1}}) \quad \text{and} \quad D_m \overset{L}{F} = (\overset{L}{F}_{,m_1} \dots \overset{L}{F}_{,m_{4(n+1)-1}}),$$

where the component partial derivatives are of the form

$$\overset{R}{F}_{,m_{4j-l}} = \begin{cases} \overset{R}{F}_{,\alpha_j} & , \; l = 3 \\ \overset{R}{F}_{,\beta_j} & , \; l = 2 \\ \overset{R}{F}_{,\rho_j} & , \; l = 1 \\ \overset{R}{F}_{,d_j} & , \; l = 0 \end{cases} \quad \text{and} \quad \overset{L}{F}_{,m_{4j-l}} = \begin{cases} 0 & , \; l = 3 \\ \overset{L}{F}_{,\beta_j} & , \; l = 2 \\ \overset{L}{F}_{,\rho_j} & , \; l = 1 \\ \overset{L}{F}_{,d_j} & , \; l = 0; \; j \neq n+1 \end{cases}$$

Consequently, according to the results in sections 6 and 7, we have at our disposal all the necessary algorithms for evaluation of (8.3).

Our final task is to present computable algorithms for the structural parameter derivative vectors of the R- and L-wave group velocities. Consequently, we have to compute the following expressions

$$D_m \overset{R}{\mathrm{v}}_g = (\overset{R}{\mathrm{v}}_{g,m_1} \dots \overset{R}{\mathrm{v}}_{g,m_{4(n+1)-1}}) \quad \text{and} \quad D_m \overset{L}{\mathrm{v}}_g = (\overset{L}{\mathrm{v}}_{g,m_1} \dots \overset{L}{\mathrm{v}}_{g,m_{4(n+1)-1}}) \tag{8.4}$$

According to Rodi *et al.* (1975) each component of (8.4) is of the form

$$\mathrm{v}_{g,m_j} = \frac{\mathrm{v}_g}{c}\left(2 - \frac{\mathrm{v}_g}{c}\right) c_{,m_j} - T\frac{\mathrm{v}_g^2}{c^2} c_{,m_jT} \tag{8.5}$$

We already have the algorithms to compute the quantities c, v_g and $c_{,m_j}$. Consequently, we need only an additional algorithm for the computation of $c_{,m_jT}$. In the present article we have achieved this numerically as follows. First the values

$$c,_{m_j}^{+} = c,_{m_j}|_{T^+} \quad \text{and} \quad c,_{m_j}^{-} = c,_{m_j}|_{T^-}$$

are formed at the points

$$T^+ = (1+\delta)T \quad \text{and} \quad T^- = (1-\delta)T,$$

respectively. In the expressions above δ is an increment of numerically small value. After these preparations we form an estimate of $c,_{m_jT}$ as the central difference approximation

$$c,_{m_jT} \simeq \frac{c,_{m_j}^{+} - c,_{m_j}^{-}}{2\delta T}$$

Under these conditions we see that (8.5) has, in both R- and L-wave cases, well defined, computable meaning.

9. *Application of the computer algorithms*

The algorithms have been prepared as a special library type subroutine package for the Burroughs B 7800 computer of the University of Helsinki. As an application of the algorithms, dispersion computations have been carried out using the structural model of Table 1, which consists of 12 layers and an underlying halfspace. The

Table 1. The structural model used in the computations.

Layer	P-wave velocity	S-wave velocity	Density	Layer thickness
1	5.90	3.33	2.65	2.00
2	6.15	3.50	2.65	1.50
3	6.00	3.40	2.60	3.00
4	6.20	3.71	2.68	13.50
5	6.50	3.54	2.74	12.00
6	7.00	4.12	2.86	3.00
7	6.95	4.09	2.85	4.00
8	7.10	4.10	2.90	8.00
9	7.60	4.39	3.00	10.00
10	8.00	4.70	3.35	13.00
11	8.30	4.80	3.40	25.00
12	8.40	4.85	3.45	25.00
13	8.50	4.91	3.50	

results of the computations have been presented in Figs. 2 to 30. The fundamental and first higher mode phase and group velocities are presented as a function of period in Figs. 2 and 3, for Rayleigh and Love waves, respectively. Figs. 4 to 30 show the partial derivatives of the dispersion quantities of Figs. 2 and 3 with respect to the structural parameters. The numbers associated with the curves refer to the layers given in Table 1. In the figures for the partial derivatives the period is the independent variable. The plotted quantities are expressed in the usual dimensions, *i.e.* velocities in (km/s), densities in (g/cm^3), layer thickness in (km) and period in (s).

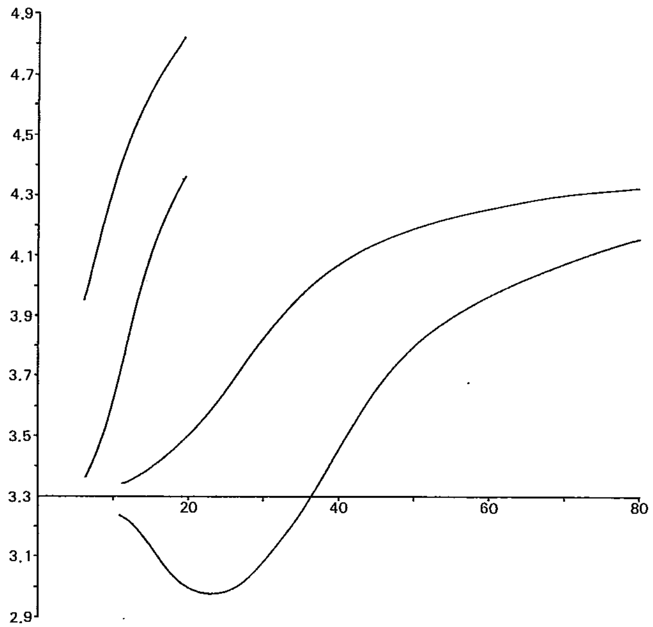

Fig. 2. Phase and group velocities of the fundamental and first higher mode Rayleigh waves.

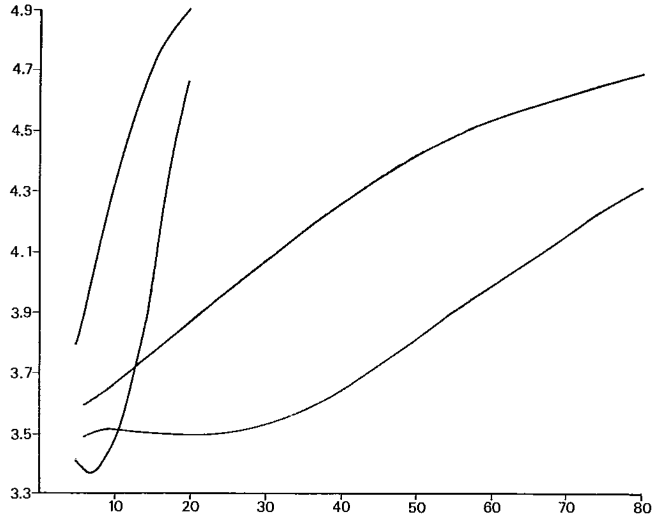

Fig. 3. Phase and group velocities of the fundamental and first higher mode Love waves.

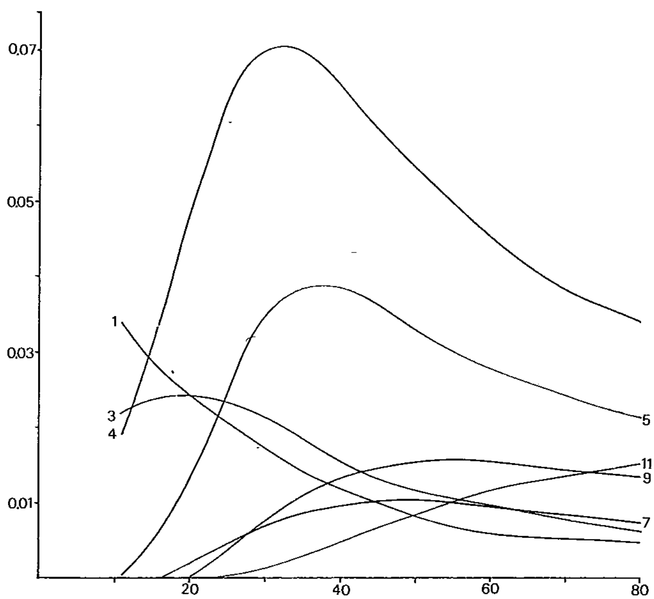

Fig. 4. Partial derivatives of the fundamental mode Rayleigh wave phase velocity with respect to the P velocities of the various layers.

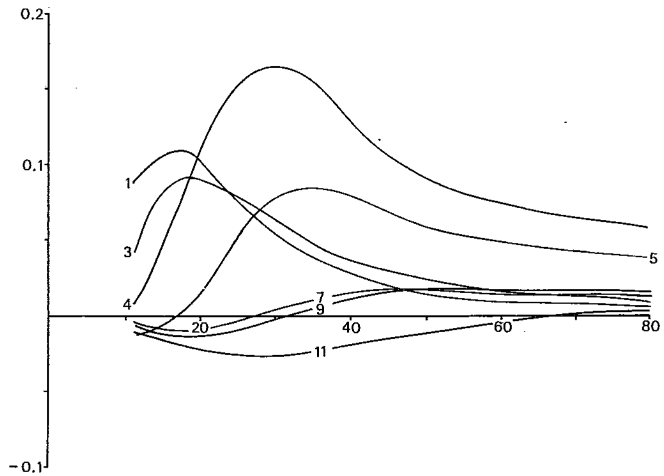

Fig. 5. Partial derivatives of the fundamental mode Rayleigh wave group velocity with respect to the P velocities of the various layers.

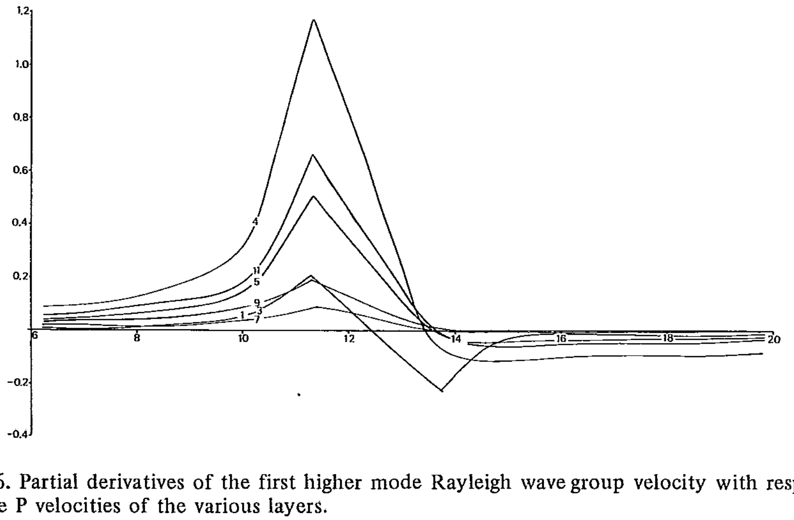

Fig. 6. Partial derivatives of the first higher mode Rayleigh wave group velocity with respect to the P velocities of the various layers.

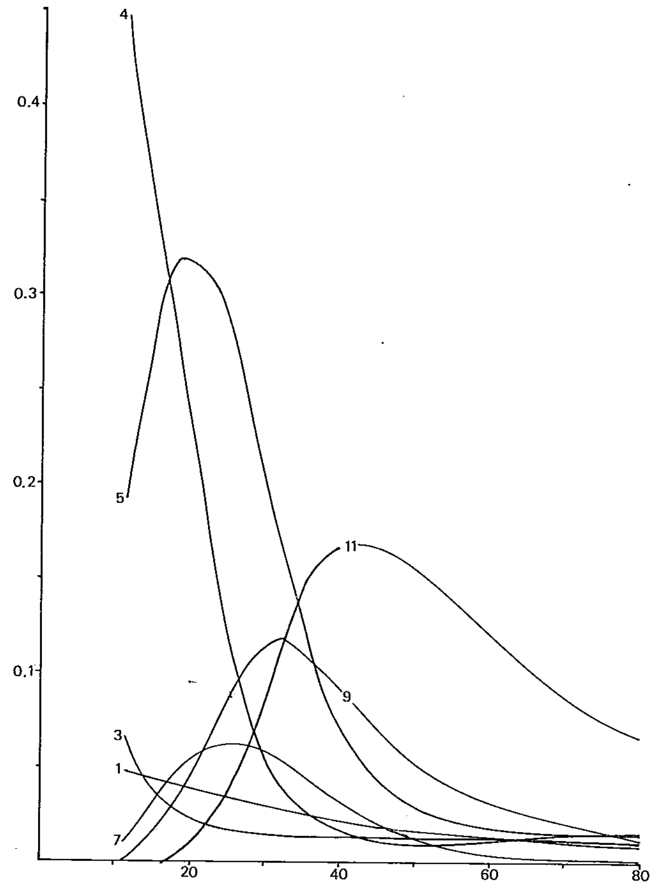

Fig. 7. Partial derivatives of the fundamental mode Rayleigh wave phase velocity with respect to the S velocities of the various layers.

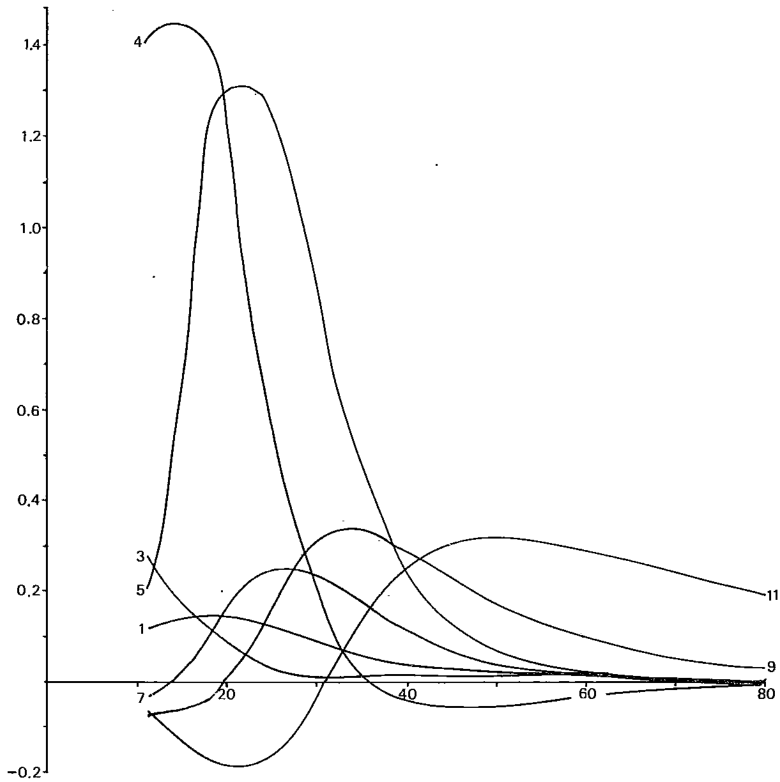

Fig. 8. Partial derivatives of the fundamental mode Rayleigh wave group velocity with respect to the S velocities of the various layers.

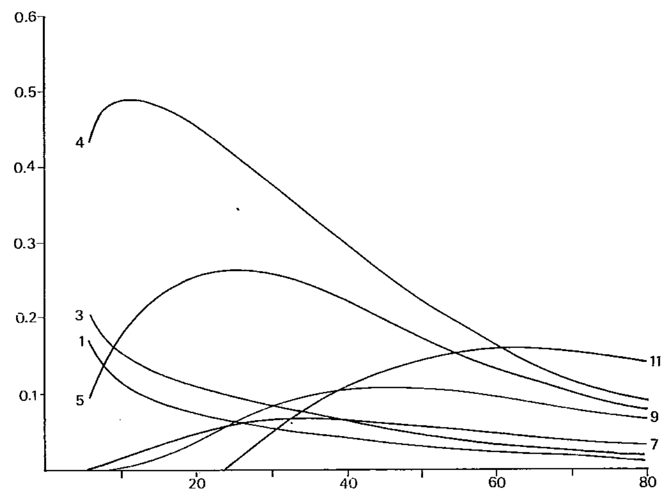

Fig. 9. Partial derivatives of the fundamental mode Love wave phase velocity with respect to the S velocities of the various layers.

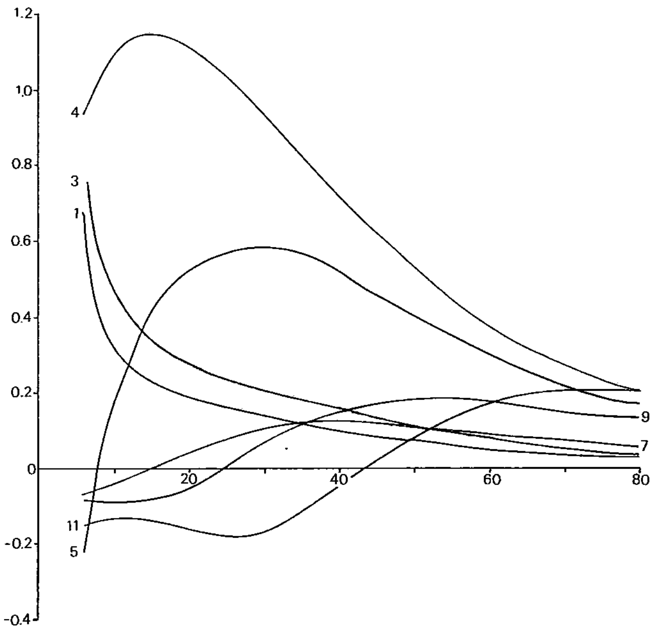

Fig. 10. Partial derivatives of the fundamental mode Love wave group velocity with respect to the S velocities of the various layers.

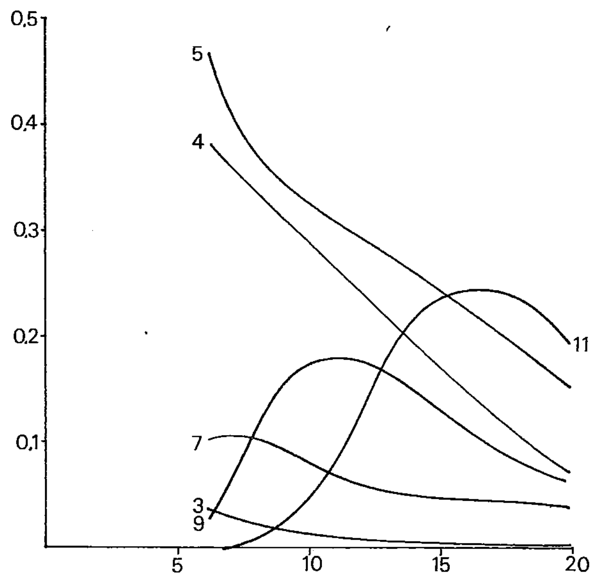

Fig. 11. Partial derivatives of the first higher mode Rayleigh wave phase velocity with respect to the S velocities of the various layers.

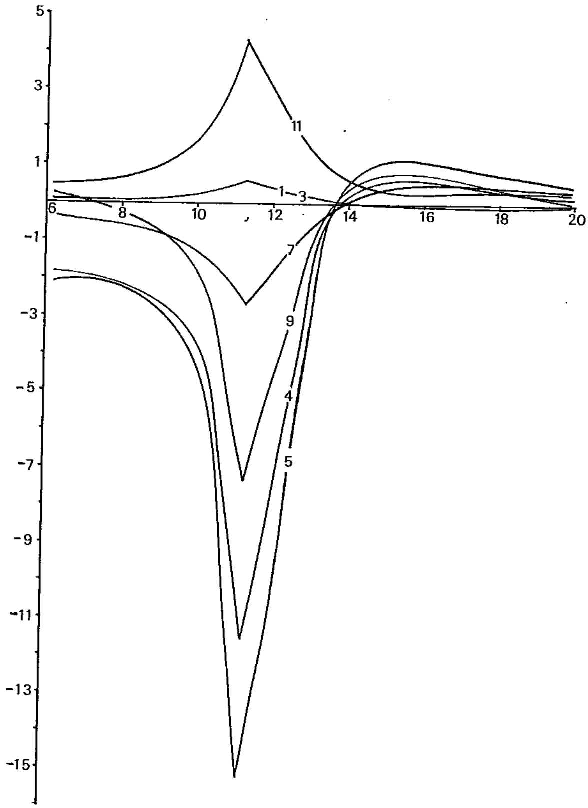

Fig. 12. Partial derivatives of the first higher mode Rayleigh wave group velocity with respect to the S velocities of the various layers.

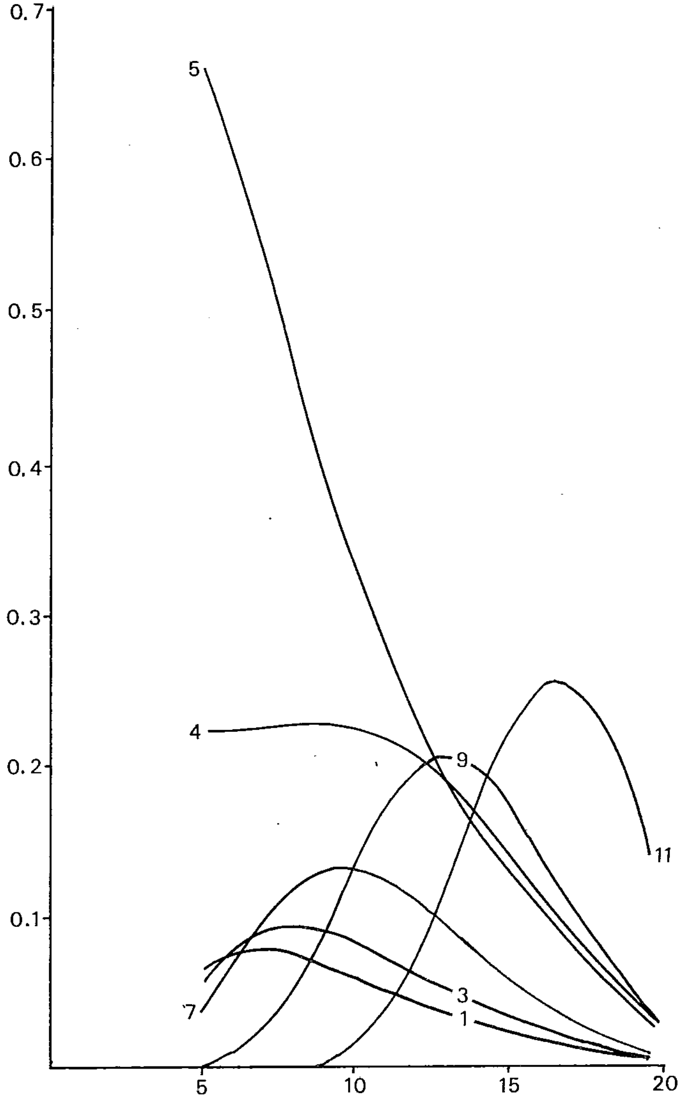

Fig. 13. Partial derivatives of the first higher mode Love wave phase velocity with respect to the S velocities of the various layers.

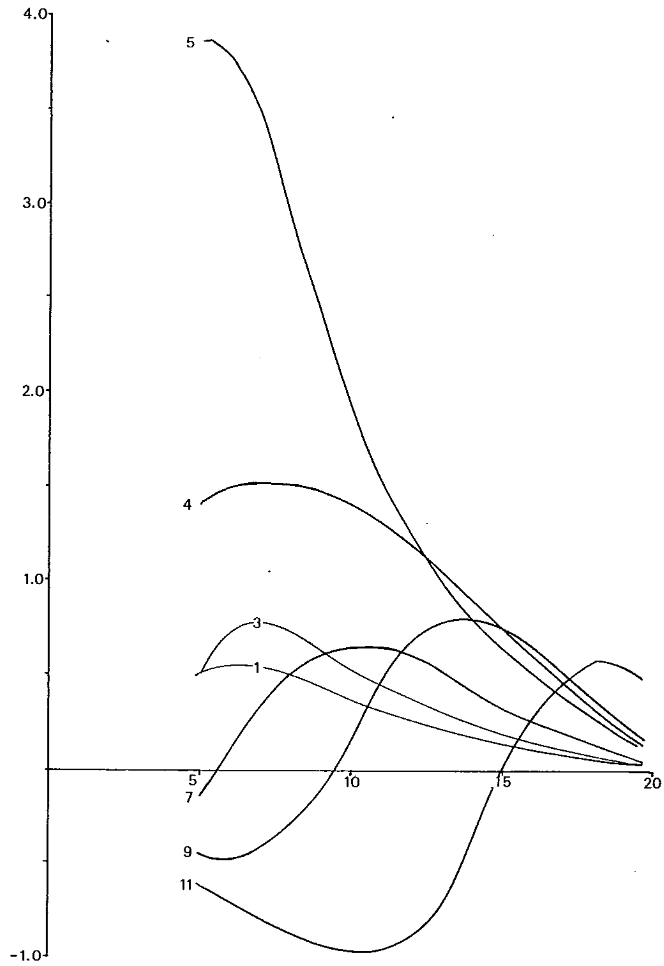

Fig. 14. Partial derivatives of the first higher mode Love wave group velocity with respect to the S velocities of the various layers.

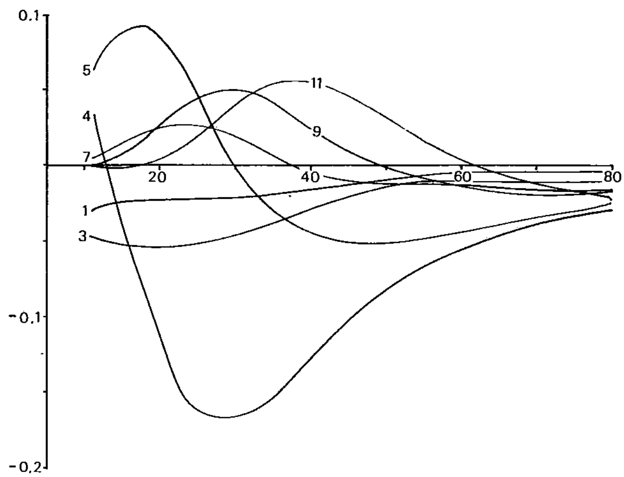

Fig. 15. Partial derivatives of the fundamental mode Rayleigh wave phase velocity with respect to the densities of the various layers.

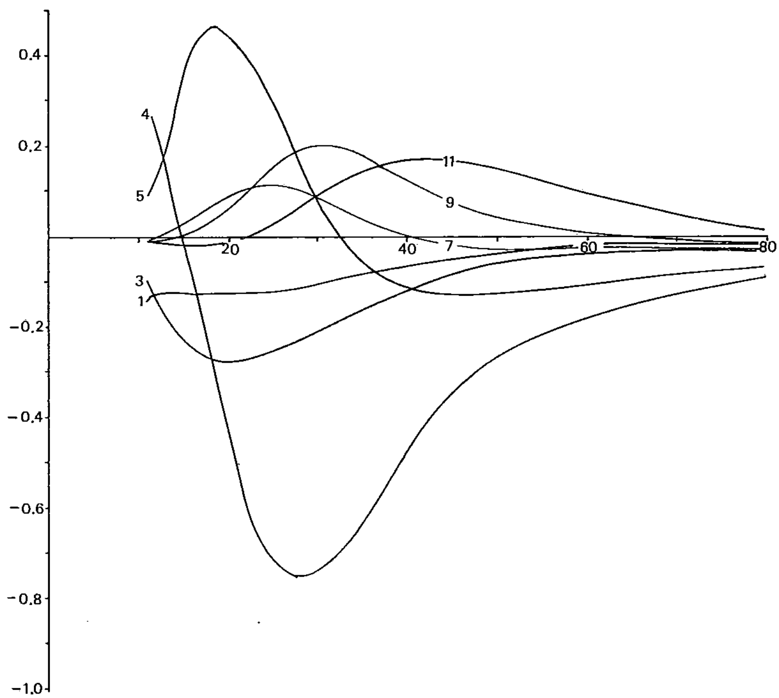

Fig. 16. Partial derivatives of the fundamental mode Rayleigh wave group velocity with respect to the densities of the various layers.

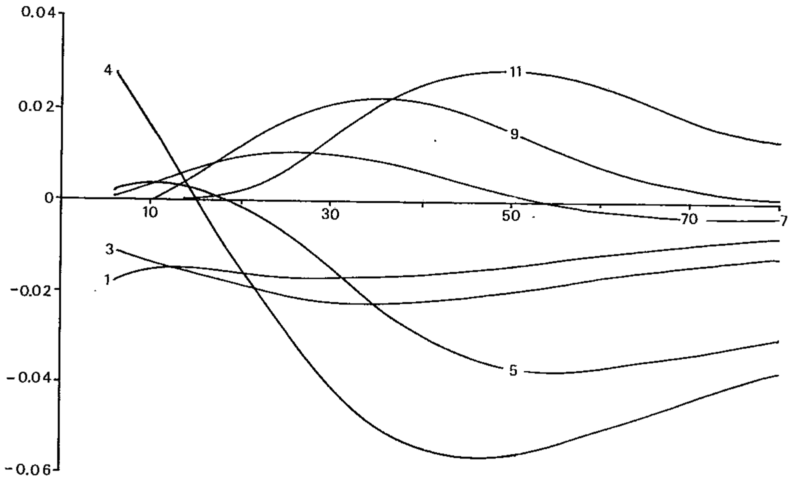

Fig. 17. Partial derivatives of the fundamental mode Love wave phase velocity with respect to the densities of the various layers.

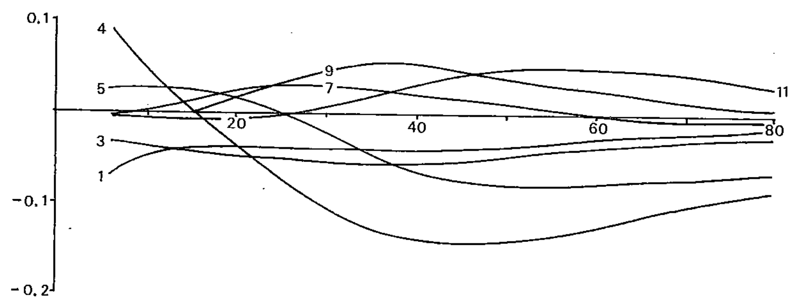

Fig. 18. Partial derivatives of the fundamental mode Love wave group velocity with respect to the densities of the various layers.

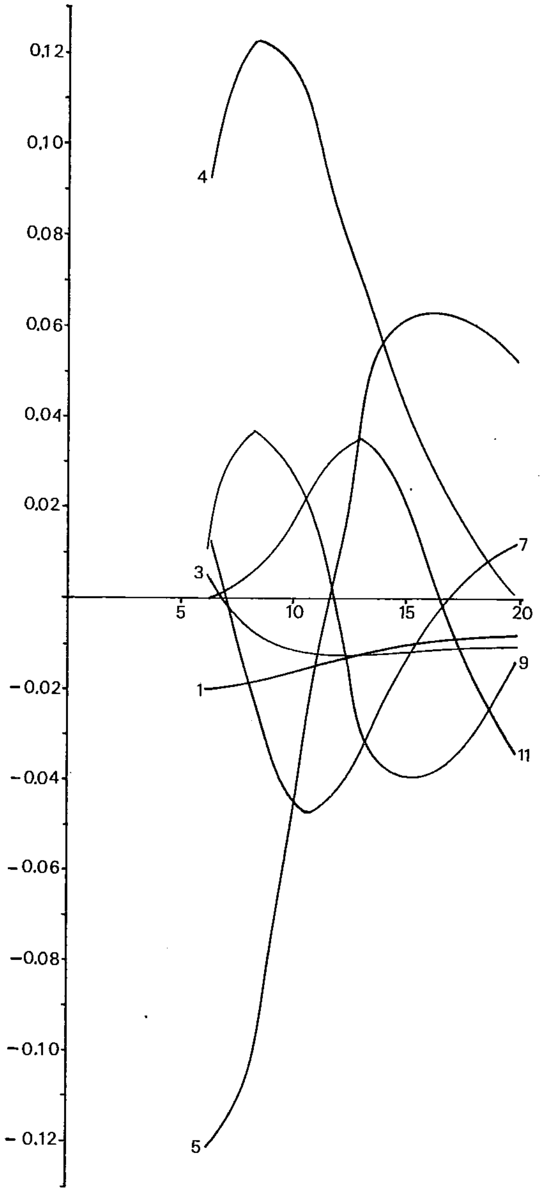

Fig. 19. Partial derivatives of the first higher mode Rayleigh wave phase velocity with respect to the densities of the various layers.

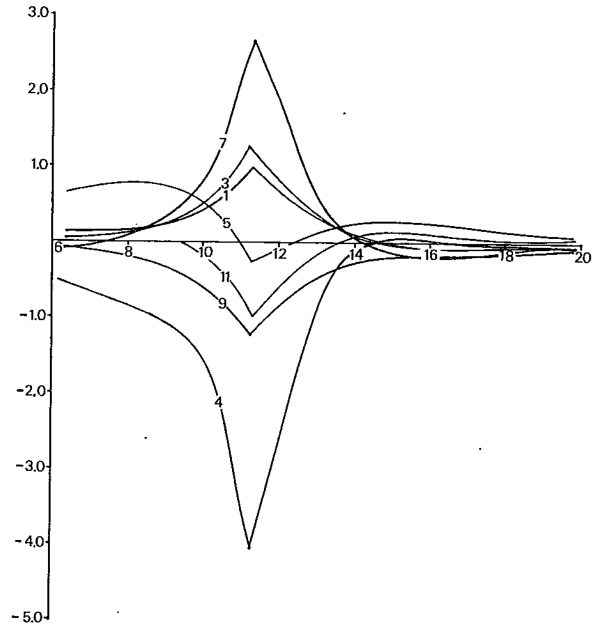

Fig. 20. Partial derivatives of the first higher mode Rayleigh wave group velocity with respect to the densities of the various layers.

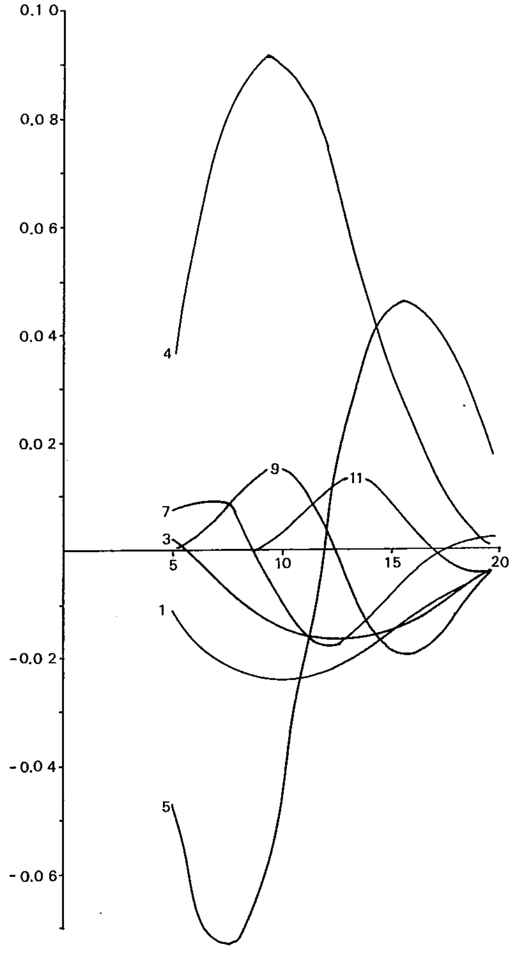

Fig. 21. Partial derivatives of the first higher mode Love wave phase velocity with respect to the densities of the various layers.

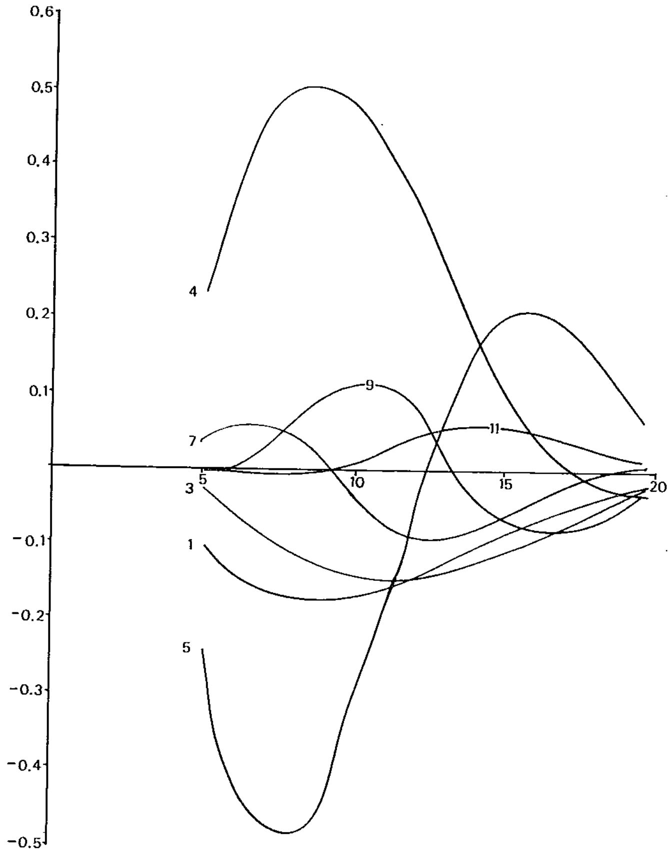

Fig. 22. Partial derivatives of the first higher mode Love group velocity with respect to the densities of the various layers.

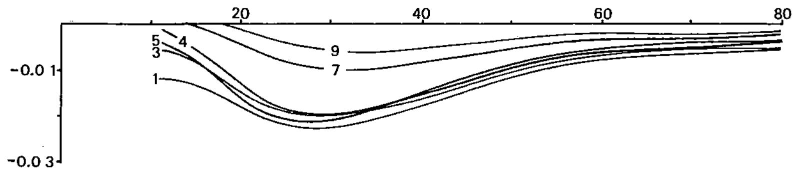

Fig. 23. Partial derivatives of the fundamental mode Rayleigh wave phase velocity with respect to the thicknesses of the various layers.

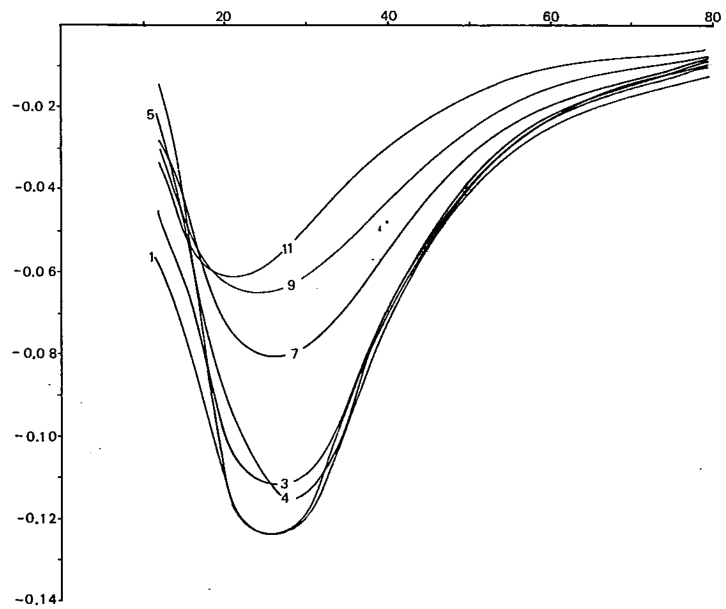

Fig. 24. Partial derivatives of the fundamental mode Rayleigh wave group velocity with respect to the thicknesses of the various layers.

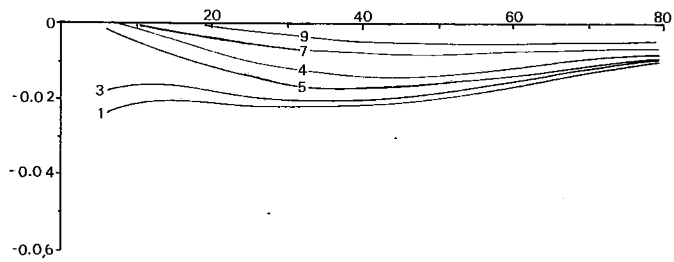

Fig. 25. Partial derivatives of the fundamental mode Love wave phase velocity with respect to the thicknesses of the various layers.

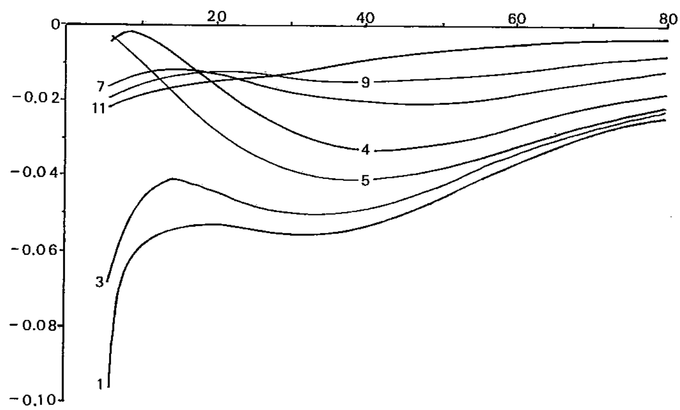

Fig. 26. Partial derivatives of the fundamental mode Love wave group velocity with respect to the thicknesses of the various layers.

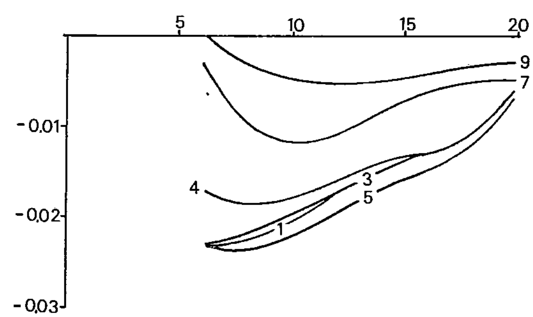

Fig. 27. Partial derivatives of the first higher mode Rayleigh wave phase velocity with respect to the thicknesses of the various layers.

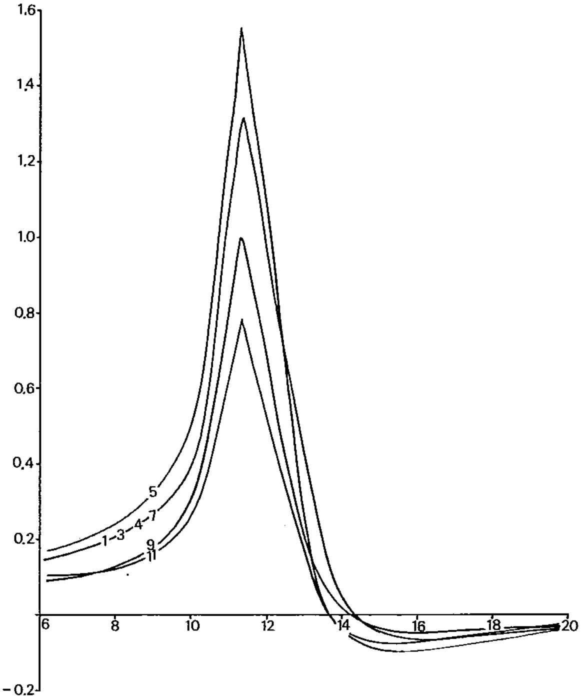

Fig. 28. Partial derivatives of the first higher mode Rayleigh wave group velocity with respect to the thicknesses of the various layers.

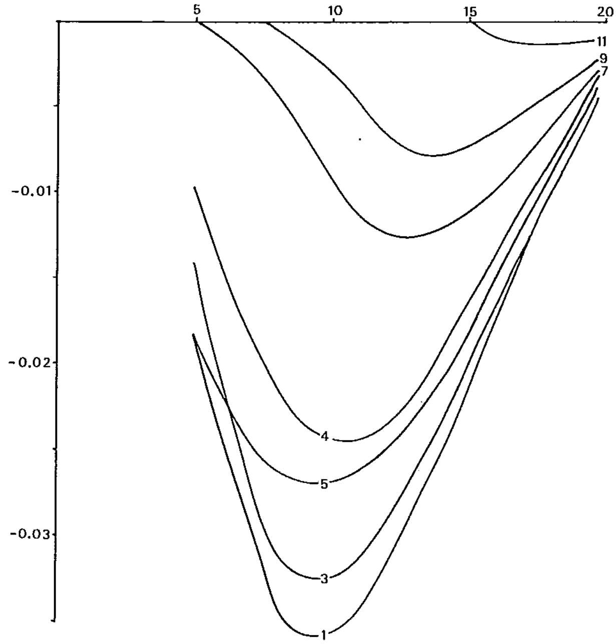

Fig. 29. Partial derivatives of the first higher mode Love wave phase velocity with respect to the thicknesses of the various layers.

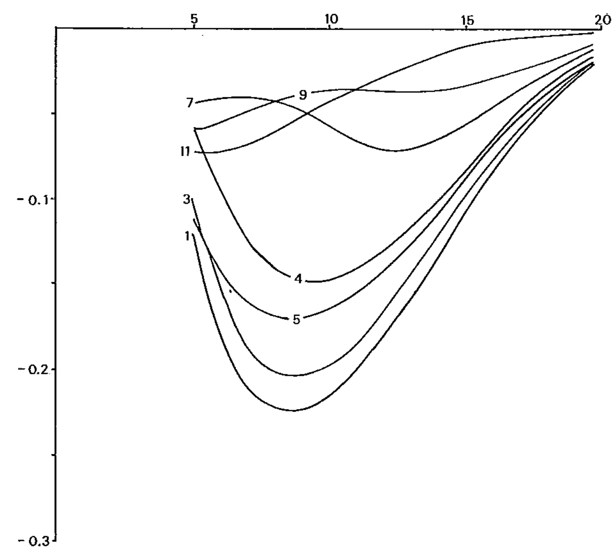

Fig. 30. Partial derivatives of the first higher mode Love wave group velocity with respect to the thicknesses of the various layers.

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APPENDIX A

1. *Eigenvectors of* $\overset{R}{A}$ *and* $\overset{R}{A}^T$

$\overset{R}{e}_1$	$\overset{R}{e}_2$	$\overset{R}{e}_3$	$\overset{R}{e}_4$
$-\frac{1}{\rho c}$	$\frac{1}{\rho c}$	$-\frac{1}{\rho c}$	$\frac{1}{\rho c}$
$\frac{1}{\rho c r_\alpha}$	$\frac{r_\beta}{\rho c}$	$-\frac{1}{\rho c r_\alpha}$	$-\frac{r_\beta}{\rho c}$
$-\frac{1-\gamma}{r_\alpha}$	γr_β	$\frac{1-\gamma}{r_\alpha}$	$-\gamma r_\beta$
γ	$1-\gamma$	γ	$1-\gamma$

$\overset{R_1}{e}$	$\overset{R_2}{e}$	$\overset{R_3}{e}$	$\overset{R_4}{e}$
$-\frac{(1-\gamma)\rho c}{2}$	$\frac{\gamma \rho c}{2}$	$-\frac{(1-\gamma)\rho c}{2}$	$\frac{\gamma \rho c}{2}$
$\frac{\gamma \rho c r_\alpha}{2}$	$\frac{(1-\gamma)\rho c}{2 r_\beta}$	$-\frac{\gamma \rho c r_\alpha}{2}$	$-\frac{(1-\gamma)\rho c}{2 r_\beta}$
$-\frac{r_\alpha}{2}$	$\frac{1}{2 r_\beta}$	$\frac{r_\alpha}{2}$	$-\frac{1}{2 r_\beta}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$$\gamma = \frac{2\beta^2}{c^2} \qquad r_\alpha = \begin{cases} \sqrt{\frac{c^2}{\alpha^2} - 1}\ ; & c \geqq \alpha \\ -i\sqrt{1 - \frac{c^2}{\alpha^2}}\ ; & c < \alpha \end{cases} \qquad r_\beta = \begin{cases} \sqrt{\frac{c^2}{\beta^2} - 1}\ ; & c \geqq \beta \\ -i\sqrt{1 - \frac{c^2}{\beta^2}}\ ; & c < \beta \end{cases}$$

2. *Eigenvectors of* $\overset{L}{A}$ *and* $\overset{L}{A}{}^{T}$

$\overset{L}{e}_1$	$\overset{L}{e}_2$
$\dfrac{c}{\rho\beta^2 r_\beta}$	$-\dfrac{c}{\rho\beta^2 r_\beta}$
1	1

$$r_\beta = \begin{cases} \sqrt{\dfrac{c^2}{\beta^2}-1}\ ; & c \geqq \beta \\ -\mathrm{i}\sqrt{1-\dfrac{c^2}{\beta^2}}\ ; & c < \beta \end{cases}$$

$\overset{L}{e}{}^{1}$	$\overset{L}{e}{}^{2}$
$\dfrac{1}{2}\dfrac{\rho\beta^2 r_\beta}{c}$	$-\dfrac{1}{2}\dfrac{\rho\beta^2 r_\beta}{c}$
$\dfrac{1}{2}$	$\dfrac{1}{2}$

3. *c-derivatives of the eigenvectors of* $\overset{R}{A}$ *and* $\overset{R}{A}^{T}$

$\overset{R}{e}_{1,c}$	$\overset{R}{e}_{2,c}$	$\overset{R}{e}_{3,c}$	$\overset{R}{e}_{4,c}$
$\frac{1}{\rho c^2}$	$-\frac{1}{\rho c^2}$	$\frac{1}{\rho c^2}$	$-\frac{1}{\rho c^2}$
$-\frac{1}{\rho c r_\alpha}\left(\frac{1}{c}+\frac{r_{\alpha,c}}{r_\alpha}\right)$	$-\frac{r_\beta}{\rho c}\left(\frac{1}{c}-\frac{r_{\beta,c}}{r_\beta}\right)$	$\frac{1}{\rho c r_\alpha}\left(\frac{1}{c}+\frac{r_{\alpha,c}}{r_\alpha}\right)$	$\frac{r_\beta}{\rho c}\left(\frac{1}{c}-\frac{r_{\beta,c}}{r_\beta}\right)$
$\frac{1-\gamma}{r_\alpha}\left(\frac{\gamma_{,c}}{1-\gamma}+\frac{r_{\alpha,c}}{r_\alpha}\right)$	$\gamma r_\beta\left(\frac{\gamma_{,c}}{\gamma}+\frac{r_{\beta,c}}{r_\beta}\right)$	$-\frac{1-\gamma}{r_\alpha}\left(\frac{\gamma_{,c}}{1-\gamma}+\frac{r_{\alpha,c}}{r_\alpha}\right)$	$-\gamma r_\beta\left(\frac{\gamma_{,c}}{\gamma}+\frac{r_{\beta,c}}{r_\beta}\right)$
$\gamma_{,c}$	$-\gamma_{,c}$	$\gamma_{,c}$	$-\gamma_{,c}$

$\overset{R_1}{e}_{,c}$	$\overset{R_2}{e}_{,c}$	$\overset{R_3}{e}_{,c}$	$\overset{R_4}{e}_{,c}$
$-\frac{(1-\gamma)\rho c}{2}\left(\frac{1}{c}-\frac{\gamma_{,c}}{1-\gamma}\right)$	$\frac{\gamma\rho c}{2}\left(\frac{1}{c}+\frac{\gamma_{,c}}{\gamma}\right)$	$-\frac{(1-\gamma)\rho c}{2}\left(\frac{1}{c}-\frac{\gamma_{,c}}{1-\gamma}\right)$	$\frac{\gamma\rho c}{2}\left(\frac{1}{c}+\frac{\gamma_{,c}}{\gamma}\right)$
$\frac{\gamma\rho c\, r_\alpha}{2}\left(\frac{1}{c}+\frac{r_{\alpha,c}}{r_\alpha}+\frac{\gamma_{,c}}{\gamma}\right)$	$\frac{(1-\gamma)\rho c}{2r_\beta}\left(\frac{1}{c}-\frac{r_{\beta,c}}{r_\beta}-\frac{\gamma_{,c}}{1-\gamma}\right)$	$-\frac{\gamma\rho c\, r_\alpha}{2}\left(\frac{1}{c}+\frac{r_{\alpha,c}}{r_\alpha}+\frac{\gamma_{,c}}{\gamma}\right)$	$-\frac{(1-\gamma)\rho c}{2r_\beta}\left(\frac{1}{c}-\frac{r_{\beta,c}}{r_\beta}-\frac{\gamma_{,c}}{1-\gamma}\right)$
$-\frac{r_\alpha}{2}\frac{r_{\alpha,c}}{r_\alpha}$	$-\frac{1}{2r_\beta}\frac{r_{\beta,c}}{r_\beta}$	$\frac{r_\alpha}{2}\frac{r_{\alpha,c}}{r_\alpha}$	$\frac{1}{2r_\beta}\frac{r_{\beta,c}}{r_\beta}$
0	0	0	0

$$\gamma_{,c} = -\frac{4\beta^2}{c^3} \qquad r_{\alpha,c} = \frac{c}{\alpha^2 r_\alpha} \qquad r_{\beta,c} = \frac{c}{\beta^2 r_\beta}$$

4. *c-derivatives of the eigenvectors of* $\overset{L}{A}$ *and* $\overset{L}{A}^T$

$\overset{L}{e}_{1,c}$	$\overset{L}{e}_{2,c}$
$\frac{c}{\rho\beta^2 r_\beta}\left(\frac{1}{c}-\frac{r_{\beta,c}}{r_\beta}\right)$	$-\frac{c}{\rho\beta^2 r_\beta}\left(\frac{1}{c}-\frac{r_{\beta,c}}{r_\beta}\right)$
0	0

$\overset{L_1}{e}_{,c}$	$\overset{L_2}{e}_{,c}$
$-\frac{1}{2}\frac{\rho\beta^2 r_\beta}{c}\left(\frac{1}{c}-\frac{r_{\beta,c}}{r_\beta}\right)$	$\frac{1}{2}\frac{\rho\beta^2 r_\beta}{c}\left(\frac{1}{c}-\frac{r_{\beta,c}}{r_\beta}\right)$
0	0

5. *α-derivatives of the eigenvectors of* $\overset{R}{A}$ *and* $\overset{R}{A}{}^T$

$\overset{R}{e}_{1,\alpha}$	$\overset{R}{e}_{2,\alpha}$	$\overset{R}{e}_{3,\alpha}$	$\overset{R}{e}_{4,\alpha}$
0	0	0	0
$-\frac{1}{\rho c r_\alpha}\frac{r_{\alpha,\alpha}}{r_\alpha}$	0	$\frac{1}{\rho c r_\alpha}\frac{r_{\alpha,\alpha}}{r_\alpha}$	0
$\frac{1-\gamma}{r_\alpha}\frac{r_{\alpha,\alpha}}{r_\alpha}$	0	$-\frac{1-\gamma}{r_\alpha}\frac{r_{\alpha,\alpha}}{r_\alpha}$	0
0	0	0	0

$\overset{R_1}{e}_{,\alpha}$	$\overset{R_2}{e}_{,\alpha}$	$\overset{R_3}{e}_{,\alpha}$	$\overset{R_4}{e}_{,\alpha}$
0	0	0	0
$\frac{\gamma\rho c r_\alpha}{2}\frac{r_{\alpha,\alpha}}{r_\alpha}$	0	$-\frac{\gamma\rho c r_\alpha}{2}\frac{r_{\alpha,\alpha}}{r_\alpha}$	0
$-\frac{r_\alpha}{2}\frac{r_{\alpha,\alpha}}{r_\alpha}$	0	$\frac{r_\alpha}{2}\frac{r_{\alpha,\alpha}}{r_\alpha}$	0
0	0	0	0

$$r_{\alpha,\alpha} = -\frac{c^2}{\alpha^3 r_\alpha}$$

6. *β-derivatives of eigenvectors of* $\overset{R}{A}$ *and* $\overset{R}{A}{}^{T}$

$\overset{R}{e}_{1,\beta}$	$\overset{R}{e}_{2,\beta}$	$\overset{R}{e}_{3,\beta}$	$\overset{R}{e}_{4,\beta}$
0	0	0	0
0	$\frac{r_\beta}{\rho c}\,\frac{r_{\beta,\beta}}{r_\beta}$	0	$-\frac{r_\beta}{\rho c}\,\frac{r_{\beta,\beta}}{r_\beta}$
$\frac{1-\gamma}{r_\alpha}\,\frac{\gamma_{,\beta}}{1-\gamma}$	$\gamma r_\beta\left(\frac{\gamma_{,\beta}}{\gamma}+\frac{r_{\beta,\beta}}{r_\beta}\right)$	$-\frac{1-\gamma}{r_\alpha}\,\frac{\gamma_{,\beta}}{1-\gamma}$	$-\gamma r_\beta\left(\frac{\gamma_{,\beta}}{\gamma}+\frac{r_{\beta,\beta}}{r_\beta}\right)$
$\gamma\,\frac{\gamma_{,\beta}}{\gamma}$	$-\gamma\,\frac{\gamma_{,\beta}}{\gamma}$	$\gamma\,\frac{\gamma_{,\beta}}{\gamma}$	$-\gamma\,\frac{\gamma_{,\beta}}{\gamma}$

$\overset{R_1}{e}_{,\beta}$	$\overset{R_2}{e}_{,\beta}$	$\overset{R_3}{e}_{,\beta}$	$\overset{R_4}{e}_{,\beta}$
$\frac{(1-\gamma)\rho c}{2}\,\frac{\gamma_{,\beta}}{1-\gamma}$	$\frac{\gamma\rho c}{2}\,\frac{\gamma_{,\beta}}{\gamma}$	$-\frac{(1-\gamma)\rho c}{2}\,\frac{\gamma_{,\beta}}{1-\gamma}$	$\frac{\gamma\rho c}{2}\,\frac{\gamma_{,\beta}}{\gamma}$
$\frac{\gamma\rho c\, r_\alpha}{2}\,\frac{\gamma_{,\beta}}{\gamma}$	$-\frac{(1-\gamma)\rho c}{2r_\beta}\left(\frac{\gamma_{,\beta}}{1-\gamma}+\frac{r_{\beta,\beta}}{r_\beta}\right)$	$-\frac{\gamma\rho c\, r_\alpha}{2}\,\frac{\gamma_{,\beta}}{\gamma}$	$\frac{(1-\gamma)\rho c}{2r_\beta}\left(\frac{\gamma_{,\beta}}{1-\gamma}+\frac{r_{\beta,\beta}}{r_\beta}\right)$
0	$-\frac{1}{2r_\beta}\,\frac{r_{\beta,\beta}}{r_\beta}$	0	$\frac{1}{2r_\beta}\,\frac{r_{\beta,\beta}}{r_\beta}$
0	0	0	0

$$\gamma_{,\beta}=\frac{4\beta}{c^2} \qquad\qquad r_{\beta,\beta}=-\frac{c^2}{\beta^3 r_\beta}$$

7. β-derivatives of eigenvectors of $\overset{L}{A}$ and $\overset{L}{A}{}^{T}$

$\overset{L}{e}_{1,\beta}$	$\overset{L}{e}_{2,\beta}$
$-\frac{c}{\rho\beta^2 r_\beta}\left(\frac{2}{\beta}+\frac{r_{\beta,\beta}}{r_\beta}\right)$	$\frac{c}{\rho\beta^2 r_\beta}\left(\frac{2}{\beta}+\frac{r_{\beta,\beta}}{r_\beta}\right)$
0	0

$\overset{L_1}{e}{}_{,\beta}$	$\overset{L_2}{e}{}_{,\beta}$
$\frac{1}{2}\frac{\rho\beta^2 r_\beta}{c}\left(\frac{2}{\beta}+\frac{r_{\beta,\beta}}{r_\beta}\right)$	$-\frac{1}{2}\frac{\rho\beta^2 r_\beta}{c}\left(\frac{2}{\beta}+\frac{r_{\beta,\beta}}{r_\beta}\right)$
0	0

8. *ρ-derivatives of eigenvectors of* $\overset{R}{A}$ *and* $\overset{R}{A}^T$

$\overset{R}{e}_{1,\rho}$	$\overset{R}{e}_{2,\rho}$	$\overset{R}{e}_{3,\rho}$	$\overset{R}{e}_{4,\rho}$
$\frac{1}{\rho^2 c}$	$-\frac{1}{\rho^2 c}$	$\frac{1}{\rho^2 c}$	$-\frac{1}{\rho^2 c}$
$-\frac{1}{\rho^2 c r_\alpha}$	$-\frac{r_\beta}{\rho^2 c}$	$\frac{1}{\rho^2 c r_\alpha}$	$\frac{r_\beta}{\rho^2 c}$
0	0	0	0
0	0	0	0

$\overset{R_1}{e}_{,\rho}$	$\overset{R_2}{e}_{,\rho}$	$\overset{R_3}{e}_{,\rho}$	$\overset{R_4}{e}_{,\rho}$
$-\frac{(1-\gamma)c}{2}$	$\frac{\gamma c}{2}$	$-\frac{(1-\gamma)c}{2}$	$\frac{\gamma c}{2}$
$\frac{\gamma c r_\alpha}{2}$	$\frac{(1-\gamma)c}{2r_\beta}$	$-\frac{\gamma c r_\alpha}{2}$	$-\frac{(1-\gamma)c}{2r_\beta}$
0	0	0	0
0	0	0	0

9. ρ-derivatives of eigenvectors of $\overset{L}{A}$ and $\overset{L}{A}^T$

$\overset{L}{e}_{1,\rho}$	$\overset{L}{e}_{2,\rho}$
$-\dfrac{c}{\rho^2\beta^2 r_\beta}$	$\dfrac{c}{\rho^2\beta^2 r_\beta}$
0	0

$\overset{L_1}{e}_{,\rho}$	$\overset{L_2}{e}_{,\rho}$
$\dfrac{1}{2}\dfrac{\beta^2 r_\beta}{c}$	$-\dfrac{1}{2}\dfrac{\beta^2 r_\beta}{c}$
0	0

APPENDIX B

1. *Operator spectral decomposition of* $\overset{R}{\mathbb{P}}$

$$\begin{aligned}\overset{R}{\mathbb{P}}\cdot Y = {} & (\overset{R}{\Pi}_1+\overset{R}{\Pi}_3)^T\cdot Y\cdot(\overset{R}{\Pi}_1+\overset{R}{\Pi}_3)+(\overset{R}{\Pi}_2+\overset{R}{\Pi}_4)^T\cdot Y\cdot(\overset{R}{\Pi}_2+\overset{R}{\Pi}_4)\\ & + qs(\overset{R}{\Pi}_1+\overset{R}{\Pi}_2)\cdot Y\cdot(\overset{R}{\Pi}_1+\overset{R}{\Pi}_2)+\frac{q}{s}(\overset{R}{\Pi}_1+\overset{R}{\Pi}_4)^T\cdot Y\cdot(\overset{R}{\Pi}_1+\overset{R}{\Pi}_4)\\ & + \frac{s}{q}(\overset{R}{\Pi}_2+\overset{R}{\Pi}_3)^T\cdot Y\cdot(\overset{R}{\Pi}_2+\overset{R}{\Pi}_3)+\frac{1}{qs}(\overset{R}{\Pi}_3+\overset{R}{\Pi}_4)^T\cdot Y\cdot(\overset{R}{\Pi}_3+\overset{R}{\Pi}_4)\end{aligned}$$

where

$$q=e^{\mathrm{i}\frac{2\pi}{T}\frac{r_\alpha}{c}d}\,,\qquad s=e^{\mathrm{i}\frac{2\pi}{T}\frac{r_\beta}{c}d}$$

$$\overset{R}{\Pi}_1=\overset{R}{e}_1\overset{R}{e}{}^1,\quad \overset{R}{\Pi}_2=\overset{R}{e}_2\overset{R}{e}{}^2,\quad \overset{R}{\Pi}_3=\overset{R}{e}_3\overset{R}{e}{}^3,\quad \overset{R}{\Pi}_4=\overset{R}{e}_4\overset{R}{e}{}^4$$

2. *Operator spectral decomposition of* $\overset{L}{\mathbb{P}}$

$$\overset{L}{\mathbb{P}}\cdot Y=s\overset{L}{\Pi}{}_1^T\cdot Y+s^{-1}\overset{L}{\Pi}{}_2^T\cdot Y$$

$$s=e^{\mathrm{i}\frac{2\pi}{T}\frac{r_\beta}{c}d}\,,\qquad \overset{L}{\Pi}{}_1^T=\overset{L}{e}{}^1\overset{L}{e}_1\,,\qquad \overset{L}{\Pi}{}_2^T=\overset{L}{e}{}^2\overset{L}{e}_2$$

3. *Operator spectral decomposition of* $\overset{R}{\mathbb{P}}_{,T}$

$$\begin{aligned}\overset{R}{\mathbb{P}}_{,T}\cdot Y = {} & (qs)_{,T}(\overset{R}{\Pi}_1+\overset{R}{\Pi}_2)^T\cdot Y\cdot(\overset{R}{\Pi}_1+\overset{R}{\Pi}_2)+\left(\frac{q}{s}\right)_{,T}(\overset{R}{\Pi}_1+\overset{R}{\Pi}_4)\cdot Y\cdot(\overset{R}{\Pi}_1+\overset{R}{\Pi}_4)\\ & +\left(\frac{s}{q}\right)_{,T}(\overset{R}{\Pi}_2+\overset{R}{\Pi}_3)\cdot Y\cdot(\overset{R}{\Pi}_2+\overset{R}{\Pi}_3)+\left(\frac{1}{qs}\right)_{,T}(\overset{R}{\Pi}_3+\overset{R}{\Pi}_4)\cdot Y\cdot(\overset{R}{\Pi}_3+\overset{R}{\Pi}_4)\end{aligned}$$

where

$$\frac{1}{qs}(qs)_{,T}=-\mathrm{i}\,\frac{2\pi}{T^2}\left(\frac{r_\alpha}{c}+\frac{r_\beta}{c}\right)d\,,\qquad \frac{s}{q}\left(\frac{q}{s}\right)_{,T}=-\mathrm{i}\,\frac{2\pi}{T^2}\left(\frac{r_\alpha}{c}-\frac{r_\beta}{c}\right)d\,,$$

$$\frac{q}{s}\left(\frac{s}{q}\right)_{,T}=\mathrm{i}\,\frac{2\pi}{T^2}\left(\frac{r_\alpha}{c}-\frac{r_\beta}{c}\right)d\,,\qquad qs\left(\frac{1}{qs}\right)_{,T}=\mathrm{i}\,\frac{2\pi}{T^2}\left(\frac{r_\alpha}{c}+\frac{r_\beta}{c}\right)d\,,$$

4. *Operator spectral decomposition of* $\overset{L}{\mathbb{P}}_{,T}$

$$\overset{L}{\mathbb{P}}_{,T} \cdot Y = s_{,T} \overset{L_T}{\Pi_1} \cdot Y + s^{-1}_{,T} \overset{L_T}{\Pi_2} \cdot Y$$

where

$$\frac{1}{s} s_{,T} = -\mathrm{i} \frac{2\pi}{T^2} \frac{r_\beta}{c} d \,, \qquad s\, s^{-1}_{,T} = \mathrm{i} \frac{2\pi}{T^2} \frac{r_\beta}{c} d$$

5. *Operator spectral decomposition of* $\overset{R}{\mathbb{P}}_{,c}$

$$\begin{aligned}
\overset{R}{\mathbb{P}}_{,c} \cdot Y = {} & (\overset{R}{\Pi}_{1,c} + \overset{R}{\Pi}_{3,c})^T \cdot Y \cdot (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_3) + (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_3)^T \cdot Y \cdot (\overset{R}{\Pi}_{1,c} + \overset{R}{\Pi}_{3,c}) \\
& + (\overset{R}{\Pi}_{2,c} + \overset{R}{\Pi}_{4,c})^T \cdot Y \cdot (\overset{R}{\Pi}_2 + \overset{R}{\Pi}_4) + (\overset{R}{\Pi}_2 + \overset{R}{\Pi}_4)^T \cdot Y \cdot (\overset{R}{\Pi}_{2,c} + \overset{R}{\Pi}_{4,c}) \\
& + qs \Big[\frac{(qs)_{,c}}{qs} (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_2)^T \cdot Y \cdot (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_2) + (\overset{R}{\Pi}_{1,c} + \overset{R}{\Pi}_{2,c})^T \cdot Y \cdot (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_2) \\
& + (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_2)^T \cdot Y \cdot (\overset{R}{\Pi}_{1,c} + \overset{R}{\Pi}_{2,c}) \Big] \\
& + \frac{q}{s} \Big[\frac{s}{q} \Big(\frac{q}{s}\Big)_{,c} (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_4)^T \cdot Y \cdot (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_4) + (\overset{R}{\Pi}_{1,c} + \overset{R}{\Pi}_{4,c})^T \cdot Y \cdot (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_4) \\
& + (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_4)^T \cdot Y \cdot (\overset{R}{\Pi}_{1,c} + \overset{R}{\Pi}_{4,c}) \Big] \\
& + \frac{s}{q} \Big[\frac{q}{s} \Big(\frac{s}{q}\Big)_{,c} (\overset{R}{\Pi}_2 + \overset{R}{\Pi}_3)^T \cdot Y \cdot (\overset{R}{\Pi}_2 + \overset{R}{\Pi}_3) + (\overset{R}{\Pi}_{2,c} + \overset{R}{\Pi}_{3,c})^T \cdot Y \cdot (\overset{R}{\Pi}_2 + \overset{R}{\Pi}_3) \\
& + (\overset{R}{\Pi}_2 + \overset{R}{\Pi}_3)^T \cdot Y \cdot (\overset{R}{\Pi}_{2,c} + \overset{R}{\Pi}_{3,c}) \Big] \\
& + \frac{1}{qs} \Big[-\frac{(qs)_{,c}}{qs} (\overset{R}{\Pi}_3 + \overset{R}{\Pi}_4)^T \cdot Y \cdot (\overset{R}{\Pi}_3 + \overset{R}{\Pi}_4) + (\overset{R}{\Pi}_{3,c} + \overset{R}{\Pi}_{4,c})^T \cdot Y \cdot (\overset{R}{\Pi}_3 + \overset{R}{\Pi}_4) \\
& + (\overset{R}{\Pi}_3 + \overset{R}{\Pi}_4)^T \cdot Y \cdot (\overset{R}{\Pi}_{3,c} + \overset{R}{\Pi}_{4,c}) \Big]
\end{aligned}$$

where

$$r_{\alpha,c} = \frac{c}{\alpha^2 r_\alpha} \quad , \qquad r_{\beta,c} = \frac{c}{\beta^2 r_\beta}$$

$$\frac{(qs)_{,s}}{qs} = \mathrm{i}\,\frac{2\pi}{T}\,\frac{d}{c}\left(r_{\alpha,c} + r_{\beta,c} - \frac{r_\alpha + r_\beta}{c}\right)$$

$$\frac{s}{q}\left(\frac{q}{s}\right)_{,c} = \mathrm{i}\,\frac{2\pi}{T}\,\frac{d}{c}\left(r_{\alpha,c} - r_{\beta,c} - \frac{r_\alpha - r_\beta}{c}\right)$$

$$\frac{q}{s}\left(\frac{s}{q}\right)_{,c} = -\frac{s}{q}\left(\frac{q}{s}\right)_{,c}$$

$$\overset{R}{\Pi}_{1,c} = \overset{R}{e}_{1,c}\,\overset{R_1}{e}{}^{1} + \overset{R}{e}_1\,\overset{R_1}{e}{}^{1}_{,c} \quad , \qquad \overset{R}{\Pi}_{2,c} = \overset{R}{e}_{2,c}\,\overset{R_2}{e}{}^{2} + \overset{R}{e}_2\,\overset{R_2}{e}{}^{2}_{,c}$$

$$\overset{R}{\Pi}_{3,c} = \overset{R}{e}_{3,c}\,\overset{R_3}{e}{}^{3} + \overset{R}{e}_3\,\overset{R_3}{e}{}^{3}_{,c} \quad , \qquad \overset{R}{\Pi}_{4,c} = \overset{R}{e}_{4,c}\,\overset{R_4}{e}{}^{4} + \overset{R}{e}_4\,\overset{R_4}{e}{}^{4}_{,c}$$

6. *Operator spectral decomposition of* $\overset{L}{\mathbb{P}}_{,c}$

$$\overset{L}{\mathbb{P}}_{,c}\cdot Y = s_{,c}\overset{L}{\Pi}{}^{T}_{1}\cdot Y + s^{-1}_{,c}\overset{L}{\Pi}{}^{T}_{2}\cdot Y + s\overset{L}{\Pi}{}^{T}_{1,c}\cdot Y + s^{-1}\overset{L}{\Pi}{}^{T}_{2,c}\cdot Y$$

where

$$s^{-1}\,s_{,c} = \mathrm{i}\,\frac{2\pi}{T}\,\frac{r_\beta}{c}\left(\frac{r_{\beta,c}}{r_\beta} - \frac{1}{c}\right)d$$

$$s\,s^{-1}_{,c} = -\,s^{-1}\,s_{,c}$$

$$\overset{L}{\Pi}{}^{T}_{1,c} = \overset{L_1}{e}{}^{1}_{,c}\,\overset{L}{e}_1 + \overset{L_1}{e}{}^{1}\,\overset{L}{e}_{1,c} \quad , \qquad \overset{L}{\Pi}{}^{T}_{2,c} = \overset{L_2}{e}{}^{2}_{,c}\,\overset{L}{e}_2 + \overset{L_2}{e}{}^{2}\,\overset{L}{e}_{2,c}$$

7. Operator spectral decomposition of $\overset{R}{\mathbb{P}}_{,\alpha}$

$$\begin{aligned}
\overset{R}{\mathbb{P}}_{,\alpha}\cdot Y =\ & (\overset{R}{\Pi}_{1,\alpha}+\overset{R}{\Pi}_{3,\alpha})^T\cdot Y\cdot(\overset{R}{\Pi}_1+\overset{R}{\Pi}_3)+(\overset{R}{\Pi}_1+\overset{R}{\Pi}_3)^T\cdot Y\cdot(\overset{R}{\Pi}_{1,\alpha}+\overset{R}{\Pi}_{3,\alpha})\\
&+(\overset{R}{\Pi}_{2,\alpha}+\overset{R}{\Pi}_{4,\alpha})^T\cdot Y\cdot(\overset{R}{\Pi}_2+\overset{R}{\Pi}_4)+(\overset{R}{\Pi}_2+\overset{R}{\Pi}_4)^T\cdot Y\cdot(\overset{R}{\Pi}_{2,\alpha}+\overset{R}{\Pi}_{4,\alpha})\\
&+qs\Bigg[\frac{(qs)_{,\alpha}}{qs}(\overset{R}{\Pi}_1+\overset{R}{\Pi}_2)^T\cdot Y\cdot(\overset{R}{\Pi}_1+\overset{R}{\Pi}_2)+(\overset{R}{\Pi}_{1,\alpha}+\overset{R}{\Pi}_{2,\alpha})^T\cdot Y\cdot(\overset{R}{\Pi}_1+\overset{R}{\Pi}_2)\\
&+(\overset{R}{\Pi}_1+\overset{R}{\Pi}_2)^T\cdot Y\cdot(\overset{R}{\Pi}_{1,\alpha}+\overset{R}{\Pi}_{2,\alpha})\Bigg]\\
&+\frac{q}{s}\Bigg[\frac{s}{q}\left(\frac{q}{s}\right)_{,\alpha}(\overset{R}{\Pi}_1+\overset{R}{\Pi}_4)^T\cdot Y\cdot(\overset{R}{\Pi}_1+\overset{R}{\Pi}_4)+(\overset{R}{\Pi}_{1,\alpha}+\overset{R}{\Pi}_{4,\alpha})^T\cdot Y\cdot(\overset{R}{\Pi}_1+\overset{R}{\Pi}_4)\\
&+(\overset{R}{\Pi}_1+\overset{R}{\Pi}_4)^T\cdot Y\cdot(\overset{R}{\Pi}_{1,\alpha}+\overset{R}{\Pi}_{4,\alpha})\Bigg]\\
&+\frac{s}{q}\Bigg[\frac{q}{s}\left(\frac{s}{q}\right)_{,\alpha}(\overset{R}{\Pi}_2+\overset{R}{\Pi}_3)^T\cdot Y\cdot(\overset{R}{\Pi}_2+\overset{R}{\Pi}_3)+(\overset{R}{\Pi}_{2,\alpha}+\overset{R}{\Pi}_{3,\alpha})^T\cdot Y\cdot(\overset{R}{\Pi}_2+\overset{R}{\Pi}_3)\\
&+(\overset{R}{\Pi}_2+\overset{R}{\Pi}_3)^T\cdot Y\cdot(\overset{R}{\Pi}_{2,\alpha}+\overset{R}{\Pi}_{3,\alpha})\Bigg]\\
&+\frac{1}{qs}\Bigg[-\frac{(qs)_{,\alpha}}{qs}(\overset{R}{\Pi}_3+\overset{R}{\Pi}_4)^T\cdot Y\cdot(\overset{R}{\Pi}_3+\overset{R}{\Pi}_4)+(\overset{R}{\Pi}_{3,\alpha}+\overset{R}{\Pi}_{4,\alpha})^T\cdot Y\cdot(\overset{R}{\Pi}_3+\overset{R}{\Pi}_4)\\
&+(\overset{R}{\Pi}_3+\overset{R}{\Pi}_4)^T\cdot Y\cdot(\overset{R}{\Pi}_{3,\alpha}+\overset{R}{\Pi}_{4,\alpha})\Bigg]
\end{aligned}$$

where

$$r_{\alpha,\alpha}=\frac{c}{\alpha^3 r_\alpha}\ , \qquad \frac{(qs)_{,\alpha}}{qs}=\mathrm{i}\,\frac{2\pi}{T}\,\frac{d}{c}\,r_{\alpha,\alpha},$$

$$\frac{s}{q}\left(\frac{q}{s}\right)_{,\alpha}=\mathrm{i}\,\frac{2\pi}{T}\,\frac{d}{c}\,r_{\alpha,\alpha},\quad \frac{q}{s}\left(\frac{s}{q}\right)_{,\alpha}=-\mathrm{i}\,\frac{2\pi}{T}\,\frac{d}{c}\,r_{\alpha,\alpha},$$

$$\overset{R}{\Pi}_{1,\alpha}=\overset{R}{e}_{1,\alpha}\overset{R}{e}{}^1+\overset{R}{e}_1\overset{R}{e}{}^1_{,\alpha}\ ,\qquad \overset{R}{\Pi}_{2,\alpha}=\overset{R}{e}_{2,\alpha}\overset{R}{e}{}^2+\overset{R}{e}_2\overset{R}{e}{}^2_{,\alpha}$$

$$\overset{R}{\Pi}_{3,\alpha}=\overset{R}{e}_{3,\alpha}\overset{R}{e}{}^3+\overset{R}{e}_3\overset{R}{e}{}^3_{,\alpha}\ ,\qquad \overset{R}{\Pi}_{4,\alpha}=\overset{R}{e}_{4,\alpha}\overset{R}{e}{}^4+\overset{R}{e}_4\overset{R}{e}{}^4_{,\alpha}$$

8. *Operator spectral decomposition of* $\overset{R}{\mathbb{P}}_{,\beta}$

$$
\begin{aligned}
\overset{R}{\mathbb{P}}_{,\beta}\cdot Y = {} & (\overset{R}{\Pi}_{1,\beta}+\overset{R}{\Pi}_{3,\beta})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{1}+\overset{R}{\Pi}_{3}) + (\overset{R}{\Pi}_{1}+\overset{R}{\Pi}_{3})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{1,\beta}+\overset{R}{\Pi}_{3,\beta}) \\
& + (\overset{R}{\Pi}_{2,\beta}+\overset{R}{\Pi}_{4,\beta})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{2}+\overset{R}{\Pi}_{4}) + (\overset{R}{\Pi}_{2}+\overset{R}{\Pi}_{4})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{2,\beta}+\overset{R}{\Pi}_{4,\beta}) \\
& + qs\Bigg[\frac{(qs)_{,\beta}}{qs}(\overset{R}{\Pi}_{1}+\overset{R}{\Pi}_{2})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{1}+\overset{R}{\Pi}_{2}) + (\overset{R}{\Pi}_{1,\beta}+\overset{R}{\Pi}_{2,\beta})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{1}+\overset{R}{\Pi}_{2}) \\
& + (\overset{R}{\Pi}_{1}+\overset{R}{\Pi}_{2})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{1,\beta}+\overset{R}{\Pi}_{2,\beta})\Bigg] \\
& + \frac{q}{s}\Bigg[\frac{s}{q}\left(\frac{q}{s}\right)_{,\beta}(\overset{R}{\Pi}_{1}+\overset{R}{\Pi}_{4})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{1}+\overset{R}{\Pi}_{4}) + (\overset{R}{\Pi}_{1,\beta}+\overset{R}{\Pi}_{4,\beta})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{1}+\overset{R}{\Pi}_{4}) \\
& + (\overset{R}{\Pi}_{1}+\overset{R}{\Pi}_{4})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{1,\beta}+\overset{R}{\Pi}_{4,\beta})\Bigg] \\
& + \frac{s}{q}\Bigg[\frac{q}{s}\left(\frac{s}{q}\right)_{,\beta}(\overset{R}{\Pi}_{2}+\overset{R}{\Pi}_{3})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{2}+\overset{R}{\Pi}_{3}) + (\overset{R}{\Pi}_{2,\beta}+\overset{R}{\Pi}_{3,\beta})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{2}+\overset{R}{\Pi}_{3}) \\
& + (\overset{R}{\Pi}_{2}+\overset{R}{\Pi}_{3})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{2,\beta}+\overset{R}{\Pi}_{3,\beta})\Bigg] \\
& + \frac{1}{qs}\Bigg[-\frac{(qs)_{,\beta}}{qs}(\overset{R}{\Pi}_{3}+\overset{R}{\Pi}_{4})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{3}+\overset{R}{\Pi}_{4}) + (\overset{R}{\Pi}_{3,\beta}+\overset{R}{\Pi}_{4,\beta})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{3}+\overset{R}{\Pi}_{4}) \\
& + (\overset{R}{\Pi}_{3}+\overset{R}{\Pi}_{4})^{T}\cdot Y\cdot(\overset{R}{\Pi}_{3,\beta}+\overset{R}{\Pi}_{4,\beta})\Bigg]
\end{aligned}
$$

where

$$
r_{\beta,\beta} = -\frac{c^2}{\beta^3 r_\beta}, \qquad \frac{(qs)_{,\beta}}{qs} = \mathrm{i}\,\frac{2\pi}{T}\,\frac{d}{c}\,r_{\beta,\beta},
$$

$$
\left(\frac{s}{q}\right)\left(\frac{q}{s}\right)_{,\beta} = -\mathrm{i}\,\frac{2\pi}{T}\,\frac{d}{c}\,r_{\beta,\beta}, \quad \frac{q}{s}\left(\frac{s}{q}\right)_{,\beta} = \mathrm{i}\,\frac{2\pi}{T}\,\frac{d}{c}\,r_{\beta,\beta},
$$

$$
\overset{R}{\Pi}_{1,\beta} = \overset{R}{e}_{1,\beta}\overset{R}{e}{}^{1} + \overset{R}{e}_{1}\overset{R}{e}{}^{1}_{,\beta}, \qquad \overset{R}{\Pi}_{2,\beta} = \overset{R}{e}_{2,\beta}\overset{R}{e}{}^{2} + \overset{R}{e}_{2}\overset{R}{e}{}^{2}_{,\beta}
$$

$$
\overset{R}{\Pi}_{3,\beta} = \overset{R}{e}_{3,\beta}\overset{R}{e}{}^{3} + \overset{R}{e}_{3}\overset{R}{e}{}^{3}_{,\beta}, \qquad \overset{R}{\Pi}_{4,\beta} = \overset{R}{e}_{4,\beta}\overset{R}{e}{}^{4} + \overset{R}{e}_{4}\overset{R}{e}{}^{4}_{,\beta}
$$

9. *Operator spectral decomposition of* $\overset{L}{\mathbb{P}}_{,\beta}$

$$\overset{L}{\mathbb{P}}_{,\beta}\cdot Y = s_{,\beta}\overset{L}{\Pi}{}_1^T\cdot Y + s_{,\beta}^{-1}\overset{L}{\Pi}{}_2^T\cdot Y$$
$$+ s\overset{L}{\Pi}{}_{1,\beta}^T\cdot Y + s^{-1}\overset{L}{\Pi}{}_{2,\beta}^T\cdot Y$$

where

$$s^{-1}s_{,\beta} = \mathrm{i}\,\frac{2\pi}{T}\frac{d}{c}\,r_{\beta,\beta}, \qquad s\,s_{,\beta}^{-1} = -\mathrm{i}\,\frac{2\pi}{T}\frac{d}{c}\,r_{\beta,\beta},$$

$$\overset{L}{\Pi}{}_{1,\beta}^T = \overset{L}{e}{}_{,\beta}^1\overset{L}{e}_1 + \overset{L}{e}{}^1\overset{L}{e}_{1,\beta}, \qquad \overset{L}{\Pi}{}_{1,\beta}^T = \overset{L}{e}{}_{,\beta}^1\overset{L}{e}_1 + \overset{L}{e}{}^1\overset{L}{e}_{1,\beta}$$

10. *Operator spectral decomposition of* $\overset{R}{\mathbb{P}}_{,\rho}$

$$\overset{R}{\mathbb{P}}_{,\rho}\cdot Y = (\overset{R}{\Pi}_{1,\rho}+\overset{R}{\Pi}_{3,\rho})^T\cdot Y\cdot(\overset{R}{\Pi}_1+\overset{R}{\Pi}_3) + (\overset{R}{\Pi}_1+\overset{R}{\Pi}_3)^T\cdot Y\cdot(\overset{R}{\Pi}_{1,\rho}+\overset{R}{\Pi}_{3,\rho})$$
$$+ (\overset{R}{\Pi}_{2,\rho}+\overset{R}{\Pi}_{4,\rho})^T\cdot Y\cdot(\overset{R}{\Pi}_2+\overset{R}{\Pi}_4) + (\overset{R}{\Pi}_2+\overset{R}{\Pi}_4)^T\cdot Y\cdot(\overset{R}{\Pi}_{2,\rho}+\overset{R}{\Pi}_{4,\rho})$$
$$+ qs\,[(\overset{R}{\Pi}_{1,\rho}+\overset{R}{\Pi}_{2,\rho})^T\cdot Y\cdot(\overset{R}{\Pi}_1+\overset{R}{\Pi}_2) + (\overset{R}{\Pi}_1+\overset{R}{\Pi}_2)^T\cdot Y\cdot(\overset{R}{\Pi}_{1,\rho}+\overset{R}{\Pi}_{2,\rho})]$$
$$+ \frac{q}{s}\,[(\overset{R}{\Pi}_{1,\rho}+\overset{R}{\Pi}_{4,\rho})^T\cdot Y\cdot(\overset{R}{\Pi}_1+\overset{R}{\Pi}_4) + (\overset{R}{\Pi}_1+\overset{R}{\Pi}_4)^T\cdot Y\cdot(\overset{R}{\Pi}_{1,\rho}+\overset{R}{\Pi}_{4,\rho})]$$
$$+ \frac{s}{q}\,[(\overset{R}{\Pi}_{2,\rho}+\overset{R}{\Pi}_{3,\rho})^T\cdot Y\cdot(\overset{R}{\Pi}_2+\overset{R}{\Pi}_3) + (\overset{R}{\Pi}_2+\overset{R}{\Pi}_3)^T\cdot Y\cdot(\overset{R}{\Pi}_{2,\rho}+\overset{R}{\Pi}_{3,\rho})]$$
$$+ \frac{1}{qs}\,[(\overset{R}{\Pi}_{3,\rho}+\overset{R}{\Pi}_{4,\rho})\cdot\cdot\, Y\cdot(\overset{R}{\Pi}_3+\overset{R}{\Pi}_4) + (\overset{R}{\Pi}_3+\overset{R}{\Pi}_4)^T\cdot Y\cdot(\overset{R}{\Pi}_{3,\rho}+\overset{R}{\Pi}_{4,\rho})]$$

where

$$\overset{R}{\Pi}_{1,\rho} = \overset{R}{e}_{1,\rho}\overset{R}{e}{}^1 + \overset{R}{e}_1\overset{R}{e}{}_{,\rho}^1\;, \qquad \overset{R}{\Pi}_{2,\rho} = \overset{R}{e}_{2,\rho}\overset{R}{e}{}^2 + \overset{R}{e}_2\overset{R}{e}{}_{,\rho}^2$$

$$\overset{R}{\Pi}_{3,\rho} = \overset{R}{e}_{3,\rho}\overset{R}{e}{}^3 + \overset{R}{e}_3\overset{R}{e}{}_{,\rho}^3\;, \qquad \overset{R}{\Pi}_{4,\rho} = \overset{R}{e}_{4,\rho}\overset{R}{e}{}^4 + \overset{R}{e}_4\overset{R}{e}{}_{,\rho}^4$$

11. *Operator spectral decomposition of* $\overset{L}{\mathbb{P}}_{,\rho}$

$$\overset{L}{\mathbb{P}}_{,\rho} \cdot Y = s\overset{L}{\Pi}{}^{T}_{1,\rho} \cdot Y + s^{-1}\overset{L}{\Pi}{}^{T}_{2,\rho} \cdot Y$$

where

$$\overset{L}{\Pi}{}^{T}_{1,\rho} = \overset{L}{e}{}^{1}_{,\rho}\, \overset{L}{e}_{1} + \overset{L}{e}{}^{1}\, \overset{L}{e}_{1,\rho} \quad , \qquad \overset{L}{\Pi}{}^{T}_{2,\rho} = \overset{L}{e}{}^{2}_{,\rho}\, \overset{L}{e}_{2} + \overset{L}{e}{}^{2}\, \overset{L}{e}_{2,\rho}$$

12. *Operator spectral decomposition of* $\overset{R}{\mathbb{P}}_{,d}$

$$\begin{aligned}
\overset{R}{\mathbb{P}}_{,d} \cdot Y = {} & (qs)_{,d}\, (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_2)^T \cdot Y \cdot (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_2) \\
& + \left(\frac{q}{s}\right)_{,d} (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_4)^T \cdot Y \cdot (\overset{R}{\Pi}_1 + \overset{R}{\Pi}_4) \\
& + \left(\frac{s}{q}\right)_{,d} (\overset{R}{\Pi}_2 + \overset{R}{\Pi}_3)^T \cdot Y \cdot (\overset{R}{\Pi}_2 + \overset{R}{\Pi}_3) \\
& + \left(\frac{1}{qs}\right)_{,d} (\overset{R}{\Pi}_3 + \overset{R}{\Pi}_4)^T \cdot Y \cdot (\overset{R}{\Pi}_3 + \overset{R}{\Pi}_4)
\end{aligned}$$

where

$$\frac{1}{qs}\,(qs)_{,d} = \mathrm{i}\,\frac{2\pi}{T}\,\frac{r_\alpha + r_\beta}{c} \quad , \qquad \frac{s}{q}\left(\frac{q}{s}\right)_{,d} = \mathrm{i}\,\frac{2\pi}{T}\,\frac{r_\alpha + r_\beta}{c}$$

$$\frac{q}{s}\left(\frac{s}{q}\right)_{,d} = -\,\frac{s}{q}\left(\frac{q}{s}\right)_{,d} \quad , \qquad qs\left(\frac{1}{qs}\right)_{,d} = -\,\frac{1}{qs}\,(qs)_{,d}$$

13. *Operator spectral decomposition of* $\overset{L}{\mathbb{P}}_{,d}$

$$\overset{L}{\mathbb{P}}_{,d} \cdot Y = s_{,d}\,\overset{L}{\Pi}{}^{T}_{1} \cdot Y + s^{-1}_{,d}\,\overset{L}{\Pi}{}^{T}_{2} \cdot Y$$

$$s^{-1}\, s_{,d} = \mathrm{i}\,\frac{2\pi}{T}\,\frac{r_\beta}{c} \quad , \qquad s\, s^{-1}_{,d} = -\, s^{-1}\, s_{,d}$$