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AN ITERATED TIME-INTEGRATION METHOD SUITABLE FOR FILTERING

by

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Abstract

It is suggested that averaging the result of the predictor step and the twice-iterated corrector step in the Euler-backward time integration scheme reduces truncation errors and yields strong selective damping of high frequency waves with a small phase error. The proposed scheme allows a 70 % longer time step than in the original Euler-backward scheme, which makes it as economical as the original scheme.

1. *Introduction*

Nonlinear normal mode initialization has proved to be a very effective method of initialization. Its implementation may be complicated, especially in limited area fine-mesh models, however, and so dynamic initialization with a selectively damping time-integration scheme is still a valid proposal, at least as a first aid to controlling noise when starting primitive equation integrations.

The Euler-backward (Matsuno) scheme has traditionally been used for this purpose (HALTINER and WILLIAMSON, 1980). Recently, more effective filters and modified time-integration schemes have been described, *e.g.* in DEY (1979) and MASUDA (1981). This paper documents a further modification of the damping scheme suggested in SAVIJÄRVI (1981). Its advantages include stronger damping of high-frequency (gravity) waves, better preservation of low-frequency (meteorological) waves, a very small phase error without discontinuities and the longer time step allowed. The method needs three evaluations of time derivatives per time step (instead of two as in the Matsuno scheme or one as in the non-damping leapfrog scheme), but with the longer time step allowed the efficiency of the scheme is equal to that of the original Matsuno scheme.

No numerical tests have been performed. Only error analysis and linear analysis with an oscillation equation leading to the scheme are described. Being a two-level scheme, the method is easily adapted to the adjustment part in split-explicit time integrations such as those of GADD (1978).

2. *Error analysis*

Consider a differential equation

$$\frac{dy}{dt} = f(y)$$

with a given (initial) value $y_n = y(t_n)$. The Euler-backward method for the numerical integration of this equation is the simplest of the predictor-corrector methods. First a prediction is made with a forward difference (the Euler-step):

$$y_p = y_{n+1}^0 = y_n + \Delta t \cdot f(y_n) \tag{1}$$

The value y_{n+1} obtained can be corrected by using it in a backward step:

$$y_c = y_{n+1}^1 = y_n + \Delta t \cdot f(y_{n+1}^0) \tag{2}$$

the combined scheme being called the Matsuno scheme. The iteration can be continued.

Both steps introduce a relatively large truncation error. The result from the predictor step is $y_p = y + R_p$, where y is the correct value and R_p the truncation error of the Euler method. Using the Taylor expansion, $R_p = \frac{1}{2} \Delta t^2 f''(t_p)$, where $t_n \leqslant t_p \leqslant t_{n+1}$. Similarly, after applying the corrector, the result is $y_c = y + R_c$ where R_c is the truncation error of the backward step (2); $R_c = -\frac{1}{2} \Delta t^2 f''(t_c)$; $t_n \leqslant t_c \leqslant t_{n+1}$. The difference $y_p - y_c = R_p - R_c$ is thus a measure of the error and we can write

$$R_c = \frac{R_c/R_p}{1 - R_c/R_p} (y_p - y_c) \tag{3}$$

This can be used to give an approximation for the error R_c. Assuming that $f''(t_p) \approx f''(t_c)$ we can write

$$\frac{R_c}{R_p} = \frac{-\frac{1}{2}\Delta t^2 f''(t_c)}{\frac{1}{2}\Delta t^2 f''(t_p)} = -1$$

and, from Eq. (3),

$$R_c = -\frac{1}{2}(y_p - y_c)$$

This error estimate can now be subtracted from the corrector result giving

$$y_{n+1} = y_c - R_c = y_c + \frac{1}{2}(y_p - y_c) = \frac{1}{2}(y_p + y_c) \tag{4}$$

The final value is thus the average of the forward and backward steps. In the next section this approach is applied to a wave equation to find out its stability properties.

3. *Linear oscillation equation*

Wave motion can be studied with the simple oscillation equation

$$\frac{\partial y}{\partial t} = -ikcy \tag{5}$$

where y is any variable, t the time, k the wavenumber and c the wave phase speed. Applying the Euler method to this equation gives

$$y_p = y_n - iayn \tag{6}$$

and correcting this by a backward step gives

$$y_{n+1}^1 = y_n - iay_p \tag{7}$$

where $n\Delta t$ is the time level, Δt the time step and $a = kc\Delta t$.

Introducing a wave solution $y_n = \exp(-iAn)$, where A is complex, to the difference equations (6) and (7) gives

$$e^{-iA} = 1 - a^2 - ia$$

for this Euler-backward (Matsuno) scheme. The amplitude response of a scheme is given by $AR = |e^{-iA}| = (Re^2 + Im^2)^{1/2}$ and the phase angle response normalized by the analytic solution $-a$ is given by $PR = -a^{-1}\tan^{-1}(Im/Re)$.

For the Matsuno scheme these are

$$(EB)\quad AR = (1 - a^2 + a^4)^{1/2}$$
$$PR = -a^{-1}\tan^{-1}(-a/(1-a^2))$$

indicating stability in the region $0 \leqslant a \leqslant 1$ where $AR \leqslant 1$. If we now apply the averaging of the predictor and corrector steps the result is

$$e^{-iA} = \frac{1}{2}(1 - ia + 1 - ia - a^2) = 1 - ia - \frac{1}{2}a^2$$

The amplitude response is $AR = |e^{iA}| = 1 + \frac{1}{4}a^2$

Thus the scheme is unstable, $AR > 1$, for all $a > 0$, and it is not useful for simulating wavelike phenomena. However, if another iteration with the backward corrector scheme is made, as suggested in SAVIJÄRVI (1981),

$$h^2_{n+1} = h_n - iah^1_{n+1} \tag{8}$$

after substituting the wave solution to Eqs. (6), (7) and (8) the amplitude and phase responses become:

$$(EB2)\quad AR = (1 - a^2 - a^4 + a^6)^{1/2}$$
$$PR = -a^{-1}\tan^{-1}(-a)$$

This scheme ($EB2$) is stable for $0 \leqslant a < 1.27$.

Averaging the result of the predictor step and the second iteration with the corrector gives

$$e^{iA} = 1 - \frac{1}{2}a^2 - i\left(a - \frac{1}{2}a^3\right)$$

The amplitude and phase responses are

$$(EB2A)\quad AR = \left(1 - \frac{3}{4}a^4 + \frac{1}{4}a^6\right)^{1/2}$$
$$PR = -a^{-1}\tan^{-1}(-a)$$

The amplitude responses of the schemes EB, $EB2$ and $EB2A$ are shown in Fig. 1 as a function of a. All these schemes provide damping for short waves. The averaged scheme $EB2A$ is the most scale selective and removes waves $kc\Delta t \cong 1.2$ completely. It also allows a time step that may be 70 % longer than in the Matsuno scheme EB. The slower (meteorological) waves are preserved rather well in the averaged

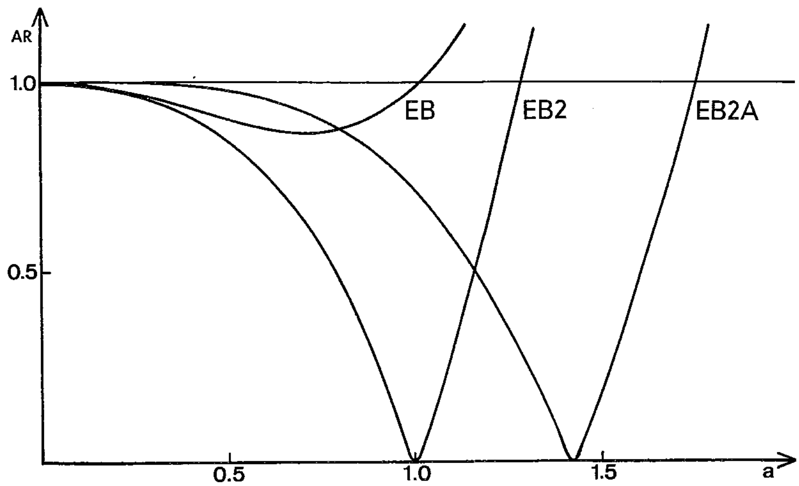

Fig. 1. Amplitude response of the Euler-backward scheme (*EB*), twicely iterated EB scheme (*EB*2) and averaged, twicely iterated EB scheme (*EB*2*A*), versus $a = kc\Delta t$.

scheme. This can be seen from Table 1, which lists the amplitude response of the three schemes for low values of *a*.

Table 1. Amplitude response as a function of *a*.

	Scheme *EB*	*EB*2	*EB*2*A*
a = 0.1	.99504	.99494	.99969
0.2	.98061	.97901	.99941
0.3	.95818	.95007	.99705
0.4	.93038	.90471	.99087
0.5	.90139	.83853	.97828
0.6	.87727	.74636	.95628

The phase speed error of the new scheme *EB*2*A* is the same as that of *EB*2, which was shown in SAVIJÄRVI (1981) (Fig. 2) to be smaller than other schemes compared. The numerical phase speed in the scheme *EB*2*A* is continuous and only very weakly dispersive throughout the stable range of wavelengths. Thus the scheme should preserve wave packet structure and physical energy dispersion properties even in a more complicated context.

4. *Conclusion*

By analysing the truncation errors of the Euler-backward time integration scheme, it was shown that averaging the results of the two steps may give a more accurate result if the second derivative of the field varies slowly. Applying this principle to the oscillation equation it was shown further that the method is unstable with only one application of the corrector step but that, with two iterations of the backward corrector, it yields a stable scheme that is strongly but selectively damping with a very small phase speed error. The method thus needs three evaluations of the derivative per time step compared with two in the Matsuno scheme, but as the time step may be 70 % longer than in the Matsuno scheme the economy of the two schemes is similar.

The time integration method proposed consists of a forward step stored temporarily, two iterations with a backward step and averaging of the final backward and temporary forward step results. As only two time levels ($n+1$ and n) are involved the scheme is self-starting and there is no computational mode. Its strongly selective filtering properties and easy implementation may be advantageous in dynamic initialization or in the gravity wave adjustment part in split explicit time integrations.

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