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# ON A MODAL APPROACH TO WAVE PROPAGATION IN A LATERALLY HOMOGENEOUS, ISOTROPIC HALFSPACE AND THE GROUP OF MOTION OF THE EUCLIDEAN PLANE

by

PEKKA R. SAASTAMOINEN

Institute of Seismology
University of Helsinki

Abstract

In this paper modal aspects to wave propagation have been treated in case of a model consisting of a laterally homogeneous, isotropic and rate-type viscoelastic halfspace, which is inhomogeneous vertically downwards. As a result from the lateral homogeneity the mode pattern describing the oscillations of the model above ought to be invariant with respect to the group of motion of the Euclidean plane. The invariance gives new possibilities in obtaining the solution of the problem.

## *Introduction*

Instead of beginning this article with a list of the many contributions to the modal treatment of seismic wave propagation in a halfspace to be found in the literature, I refer the interested reader to the classic book by EWING *et al.* [2] and to the papers by GILBERT [3] and by COCHRAN *et al.* [1] with the bibliographical references given in these works.

Here I shall apply modal analysis to the problem of seismic wave propagation in a laterally homogeneous, isotropic and rate-type viscoelastic halfspace, taking into account the interaction of the group of motion of the Euclidean plane.

To this end use has been made of the Laplace transformed equations of motion (1) and (2) presented by SAASTAMOINEN [4].

$$\varrho p^2 \mathbf{u} = \mathrm{tr}_{12}(\partial_x \mathbf{T}) + \mathbf{f} \tag{1}$$

$$\mathbf{T} = \tilde{\lambda}\,\mathrm{tr}(\partial_x \mathbf{u})\mathbf{1} + \tilde{\mu}(\partial_x \mathbf{u} + \partial_x \mathbf{u}^T) \tag{2}$$

In (1) and (2)

- $\mathbf{x} \in \Re^3$ ($\in$ means »belongs to»), where $\Re^3$ is the Euclidean three space.
- $\varrho$ , $\tilde{\lambda}$ and $\tilde{\mu} \in \Re$ are scalar fields on $\Re^3$ . Further, $\tilde{\lambda}$ and $\tilde{\mu}$ are certain polynomials in the transform parameter $p$ .
- $\mathbf{u}$ and $\mathbf{f} \in V$ are vector fields on $\Re^3$ .
- $\mathbf{T}$ and $\mathbf{1} \in L(V, V)$ are tensor fields on $\Re^3$ with $\mathbf{1}$ as the unit of $L(V, V)$ .
- $\mathrm{tr}_{12}$ means the contraction of the first two vectors following in a tensor product.

(For further details about equations (1) and (2) see the paper by SAASTAMOINEN [4].

*The modal equations*

In what follows equations (1) and (2) will be decomposed into a system of four coupled equations. Two of these have their values in the two-dimensional vector space $\hat{V}$ formed on the plane $\Re^2$ , which lies parallel to the surface of the half-space. The other two have their values in the one-dimensional vector space $V_1$ , which forms the orthogonal complement of $\hat{V}$ in $V$ . As a consequence,

$$V = V_1 \oplus \hat{V} \text{ and } L(V, V) = L(V_1 \oplus \hat{V}, V_1 \oplus \hat{V}) \tag{3}$$

where $\oplus$ refers to the orthogonal direct sum.

To achieve the decomposition described above, we take an arbitrary constant vector $\mathbf{a}$ and split it according to (3) as follows

$$\mathbf{a} = A\,\mathbf{e} + \hat{\mathbf{a}} \tag{4}$$

where $\mathbf{e}$ is the unit vector in $V_1$ , $A = (\mathbf{a}, \mathbf{e})$ $((.\,,.)$ is the scalar product in $V)$ and $\hat{\mathbf{a}}$ is the projection of $\mathbf{a}$ in $\hat{V}$ . Furthermore, the scalar product of (1) with $\mathbf{a}$ results in

$$\varrho p^2(\mathbf{u}, \mathbf{a}) = (\mathrm{tr}_{12}\partial_x \mathbf{T}, \mathbf{a}) + (\mathbf{f}, \mathbf{a}) \tag{5}$$

The different terms in (5) will be decomposed with the aid of (4). Consequently, first

$$(\mathbf{u}, \mathbf{a}) = U A + (\hat{\mathbf{u}}, \hat{\mathbf{a}}) \tag{6}$$

where $U = (\mathbf{u}, \mathbf{e})$ and $\hat{\mathbf{u}}$ is the projections of $\mathbf{u}$ in $\hat{V}$. Secondly use of the auxiliary relation

$$(\mathrm{tr}_{12}\, \partial_x \mathbf{T}, \mathbf{a}) = (\partial_x \mathbf{T}[\mathbf{a}], \mathbf{1})_{L(V,V)} \tag{7}$$

yields

$$(\mathrm{tr}_{12} \partial_x \mathbf{T}, \mathbf{a}) = A\, \partial_x P + (\partial_x \hat{\mathbf{t}}, \hat{\mathbf{a}}) + A\, \mathrm{tr}(\partial_{\hat{x}} \hat{\mathbf{t}}) + \mathrm{tr}(\partial_{\hat{x}} \hat{\mathbf{T}}[\hat{\mathbf{a}}]) \tag{8}$$

In (7) $(\cdot, \cdot)_{L(V,V)}$ is the scalar product in $L(V, V)$ (see SAASTAMOINEN [4]). In (8) $P = (\mathbf{T}[\mathbf{e}], \mathbf{e})$, $(\hat{\mathbf{t}}, \hat{\mathbf{a}}) = (\hat{\mathbf{a}}, \mathbf{T}[\mathbf{e}])$ and $(\hat{\mathbf{a}}, \hat{\mathbf{T}}[\hat{\mathbf{a}}]) = (\hat{\mathbf{a}}, \mathbf{T}[\hat{\mathbf{a}}])$. Proceeding as above we find the last term

$$(\mathbf{f}, \mathbf{a}) = FA + (\hat{\mathbf{f}}, \hat{\mathbf{a}}) \tag{9}$$

where $F = (\mathbf{t}, \mathbf{e})$ and $\hat{\mathbf{f}}$ is the projection of $\mathbf{f}$ in $\hat{V}$. As a consequence of (5), (6) (8) and (9), we obtain the decomposition of (1) given below.

$$\varrho p^2 U = \partial_x P + \mathrm{tr}(\partial_{\hat{x}} \hat{\mathbf{t}}) + F \tag{10}$$

$$\varrho p^2 \hat{\mathbf{u}} = \partial_x \hat{\mathbf{t}} + \mathrm{tr}_{12}(\partial_{\hat{x}} \hat{\mathbf{T}}) + \hat{\mathbf{f}} \tag{11}$$

Since later we use only the Fourier-transformed versions (10)′ and (11)′ of the above equation, the application of (A12) to (10) and (11) results in

$$\varrho p^2 U_k = d_x P_k + k(\hat{\mathbf{k}}^0, \hat{\mathbf{t}}_k) + F_k \tag{10)$'$}$$

$$\varrho p^2 \hat{\mathbf{u}}_k = d_x \hat{\mathbf{t}}_k + k\, \mathrm{tr}_{12}(\hat{\mathbf{k}}^0 \otimes \hat{\mathbf{T}}_k) + \hat{\mathbf{f}}_k \tag{11)$'$}$$

where quantities with the subscript $k$ belong to the space $L_k^2(C_1)$.

For the decomposition of equation (2), we need the two arbitrary constant vectors

$$\mathbf{a} = A\mathbf{e} + \hat{\mathbf{a}} \quad \text{and} \quad \mathbf{b} = B\,\mathbf{e} + \hat{\mathbf{b}} \tag{12}$$

Thereafter we form the bilinear products

$$(\mathbf{T}[\mathbf{a}], \mathbf{b}) = ABP + A(\hat{\mathbf{t}}, \hat{\mathbf{b}}) + B(\hat{\mathbf{a}}, \hat{\mathbf{t}}) + (\hat{\mathbf{T}}[\hat{\mathbf{a}}], \hat{\mathbf{b}}) \tag{13}$$

and

$$(\mathbf{T}[\mathbf{a}], \mathbf{b}] = \tilde{\lambda}(\mathrm{tr}(\partial_x \mathbf{u})\, \mathbf{1}\, [\mathbf{a}], \mathbf{b}) + \tilde{\mu}((\partial_x \mathbf{u} + \partial_x \mathbf{u}^T)[\mathbf{a}], \mathbf{b}) \tag{14}$$

where

$$\mathrm{tr}(\partial_x \mathbf{u}) = \partial_x U + \mathrm{tr}(\partial_{\hat{x}} \hat{\mathbf{u}}) \tag{15}$$

As to the individual terms in (14), the use of (12) and (15) shows that

$$(\mathrm{tr}(\partial_x \mathbf{u})\ \mathbf{1}\ [\mathbf{a}]\ ,\ \mathbf{b}) = AB(\partial\ U + \mathrm{tr}(\partial_{\hat{x}}\hat{\mathbf{u}}) + ((\partial_x U + \mathrm{tr}(\partial_{\hat{x}}\ \hat{\mathbf{u}}))\ \hat{\mathbf{a}}\ ,\ \hat{\mathbf{b}}) \tag{16}$$

and

$$\begin{aligned}((\partial_x \mathbf{u} + \partial_x \mathbf{u}^T)\ [\mathbf{a}]\ ,\ \mathbf{b}) = 2AB\partial_x U + A(\partial_x \hat{\mathbf{u}} + \partial_{\hat{x}} U\ ,\ \hat{\mathbf{b}}) \\ + B(\hat{\mathbf{a}}\ ,\ \partial_{\hat{x}} U + \partial_x \hat{\mathbf{u}}) + ((\partial_{\hat{x}}\hat{\mathbf{u}} + \partial_{\hat{x}}\hat{\mathbf{u}}^T)\ [\hat{\mathbf{a}}]\ ,\ \hat{\mathbf{b}})\end{aligned} \tag{17}$$

Finally, comparison of (14) with (13) (supplemented with (16) and (17) yields the decomposition of equation (2) given below

$$P = (\tilde{\lambda} + 2\tilde{\mu})\ \partial_x U + \tilde{\lambda}\ \mathrm{tr}(\partial_{\hat{x}}\hat{\mathbf{u}}) \tag{18}$$

$$\hat{\mathbf{t}} = \tilde{\mu}(\partial_x \hat{\mathbf{u}} + \partial_{\hat{x}} U) \tag{19}$$

and

$$\hat{\mathbf{T}} = \tilde{\lambda}\partial_x U\ \hat{\mathbf{1}} + \tilde{\lambda}\ \mathrm{tr}(\partial_{\hat{x}}\hat{\mathbf{u}})\ \hat{\mathbf{1}} + \tilde{\mu}\ (\partial_{\hat{x}}\hat{\mathbf{u}} + \partial_{\hat{x}}\hat{\mathbf{u}}^T) \tag{20}$$

As before the application of (A12) to the equations above results in the Fourier-transformed versions

$$P_k = (\tilde{\lambda} + 2\tilde{\mu})\ d_x U_k + \tilde{\lambda}\ k(\hat{\mathbf{k}}^0\ ,\ \hat{\mathbf{u}}_k) \tag{18$'$}$$

$$\hat{\mathbf{t}}_k = \tilde{\mu}\ d_x \hat{\mathbf{u}}_k + \tilde{\mu} k \hat{\mathbf{k}}^0 U_k \tag{19$'$}$$

and

$$\hat{T}_k = \tilde{\lambda} d_x U_k\ \hat{\mathbf{1}} + \tilde{\lambda} k(\hat{\mathbf{k}}^0\ ,\ \hat{\mathbf{u}}_k)\ \hat{\mathbf{1}} + \tilde{\mu}\ k(\hat{\mathbf{k}}^0 \otimes \hat{\mathbf{u}}_k + \hat{\mathbf{u}}_k \otimes \hat{\mathbf{k}}^0) \tag{20$'$}$$

Equations (10)′, (11)′, (18)′, (19)′ and (20)′, however, contain the term $\hat{\mathbf{T}}_k$, which is not compatible with the boundary conditions. To eliminate it later we form from (20)′ and (18)′ relation

$$\begin{aligned}\hat{\mathbf{k}}^0 \otimes \hat{\mathbf{T}}_k = \frac{\tilde{\lambda}}{\tilde{\lambda} + 2\tilde{\mu}} P_k \hat{\mathbf{k}}^0 \otimes \hat{\mathbf{1}} + \frac{2\tilde{\lambda}\tilde{\mu}k}{\tilde{\lambda} + 2\tilde{\mu}}\ (\hat{\mathbf{k}}^0\ ,\ \hat{\mathbf{u}}_k)\hat{\mathbf{k}}^0 \otimes \hat{\mathbf{1}} + \\ \tilde{\mu}k(\hat{\mathbf{k}}^0 \otimes \hat{\mathbf{k}}^0 \otimes \hat{\mathbf{u}}_k + \hat{\mathbf{k}}^0 \otimes \hat{\mathbf{u}} \otimes \hat{\mathbf{k}}^0)\end{aligned} \tag{20$''$}$$

*Radial and transverse parts of the modal equations*

When the system (10)′, (11)′ (supplemented with (20)″), (18)′ and (19)′ is projected in the two orthogonal directions $\hat{\mathbf{k}}^0$ and $\hat{\mathbf{k}}^0_\perp((\hat{\mathbf{k}}^0\ ,\hat{\mathbf{k}}^0_\perp) = 0$, in cases of lateral homogeneity, $\hat{\mathbf{k}}^0$ and $\hat{\mathbf{k}}^0_\perp$ correspond to the radial

and transverse directions), we obtain the two useful systems of equations.
— The radial part

$$
\begin{aligned}
d_x U_k &= -\frac{\tilde{\lambda} k}{\tilde{\lambda} + 2\tilde{\mu}} V_k + \frac{1}{\tilde{\lambda} + 2\tilde{\mu}} P_k \\
d_x V_k &= -U_k + \frac{1}{\tilde{\mu}} R_k \\
d_x P_k &= \varrho p^2 U_k - k R_k - F_k \\
d_x R_k &= \left(\varrho p^2 - \frac{4\tilde{\mu}(\tilde{\lambda} + \tilde{\mu})\, k^2}{\tilde{\lambda} + 2\tilde{\mu}}\right) V_k - \frac{\tilde{\lambda} k}{\tilde{\lambda} + 2\tilde{\mu}} P_k - \underset{G}{F_k}
\end{aligned} \tag{21}
$$

where $V_k = (\hat{\mathbf{u}}_k, \hat{\mathbf{k}}^0)$, $R_k = (\hat{\mathbf{t}}_k, \hat{\mathbf{k}}^0)$ and $\underset{G}{F_k} = (\hat{\mathbf{f}}_k, \hat{\mathbf{k}}^0)$. (The other quantities have been defined previously.)
— The transverse part

$$
\begin{aligned}
d_x W_k &= \frac{1}{\tilde{\mu}} S_k \\
d_x S_k &= (\varrho p^2 - \tilde{\mu} k^2) W_k - \underset{R}{F_k}
\end{aligned} \tag{22}
$$

where $W_k = (\hat{\mathbf{u}}_k, \hat{\mathbf{k}}_\perp^0)$, $S_k = (\hat{\mathbf{t}}_k, \hat{\mathbf{k}}_\perp^0)$ and $\underset{R}{F_k} = (\hat{\mathbf{f}}_k, \hat{\mathbf{k}}_\perp^0)$.

To simplify subsequent manipulations we present (21) and (22) in the compact form

$$
d_x y_k = A_k y_k + f_k \tag{23}
$$

In (23) $y_k$ and $t_k \in \mathcal{L}_k^2(C_1)$, $A_k \in L(\mathcal{L}_k^2(C_1), \mathcal{L}_k^2(C_1))$ (the space linear transformations on $\mathcal{L}_k^2(C_1)$). Further
— The radial part

$$
y_k = \begin{vmatrix} U_k \\ V_k \\ P_k \\ R_k \end{vmatrix}, \quad f_k = \begin{vmatrix} 0 \\ 0 \\ -F_k \\ -\underset{G}{F_k} \end{vmatrix} \quad \text{and } A_k = \begin{vmatrix} 0 & \dfrac{\tilde{\lambda} k^2}{\tilde{\lambda} + 2\tilde{\mu}} & \dfrac{1}{\lambda + 2\tilde{\mu}} & 0 \\ -1 & 0 & 0 & \dfrac{1}{\tilde{\mu}} \\ \varrho p^2 & 0 & 0 & k^2 \\ 0 & \varrho p^2 - \dfrac{4\tilde{\mu}(\tilde{\lambda} + \tilde{\mu})k^2}{\tilde{\lambda} + 2\tilde{\mu}} & -\dfrac{\tilde{\lambda}}{\tilde{\lambda} + 2\tilde{\mu}} & 0 \end{vmatrix} \tag{24}
$$

— The transverse part

$$y_k = \begin{pmatrix} W_k \\ S_k \end{pmatrix}, \; f_k = \begin{pmatrix} 0 \\ -F_k \\ {\scriptstyle R} \end{pmatrix} \text{ and } A_k = \begin{pmatrix} 0 & \frac{1}{\tilde{\mu}} \\ \varrho p^2 + \tilde{\mu} k^2 & 0 \end{pmatrix} \tag{25}$$

*The boundary conditions*

The boundary conditions as usual express the vanishing of the stresses at the free surface $(x = 0)$ and the vanishing of the displacements below some reference depth $(x = x_\infty)$. In other words,

$$W_0 y_k(o) + W_\infty y_k(x_\infty) = 0 \tag{26}$$

where

— For the partial part

$$W_0 = \begin{pmatrix} 0 & 0 \\ 0 & \hat{I} \end{pmatrix} \text{ and } W_\infty = \begin{pmatrix} \hat{I} & 0 \\ 0 & 0 \end{pmatrix} \tag{27}$$

where $\hat{I}$ is the $2 \times 2$ unit matrix.

— For the transverse part

$$W_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } W_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{28}$$

*Solution of the problem (23) and (26)*

Since the solution of a similar boundary value problem was treated in greater detail in the paper by SAASTAMOINEN [4], only some trends will be given here.

According to the paper mentioned above, the solution of (23) and (26) shows the following time depence

$$y_k(x, t) = \frac{1}{2\pi i} \int\limits_{LC} e^{pt} \left( \int\limits_0^{x_\infty} G_k(p, x, \xi) f_k(p, \xi) \, d\xi \right) dp \tag{29}$$

where $LC$ is the Laplace contour and

$$G_k(p, x, \xi) = \begin{cases} Y_k(p, x) D_k^{-1}(p) W_0 Y_k^{-1}(p, \xi); & \xi \leq x \\ -Y_k(p, x) D_k^{-1}(p) W_\infty Y_k^{-1}(p, \xi); & \xi \geq x \end{cases} \tag{30}$$

The operator $Y_k(p, \xi)$ in (30) is the solution to the initial value problem

$$d_x Y_k = A_k Y_k\,;\;\; Y_k(0) = I \tag{31}$$

where $I$ is the unit of $L(\mathscr{L}_k^2(C_1)\,,\,\mathscr{L}_k^2(C_1))$ . Further,

$$D_k(p) = W_0 + W_\infty Y_k(x_\infty) \tag{32}$$

is the characteristic matrix of the problem.

The evaluation of (29) with respect to the contributions from the poles on the different Riemann sheets of

$$\det D_k(p) = 0$$

results in the long-time approximation

$$y_k(x\,,\,t) \cong \sum_j e^{p_j(k)t} \int Y_k(p_j(k)\,,\,x) \operatorname*{Res}_{p=p_j(k)} R_k(p) Y_k^{-1}(p_j(k)\,,\,\xi) f_k(p_j(k)\,,\,\xi) d\xi \tag{33}$$

where

$$\operatorname*{Res}_{p=p_j(k)} R_k(p) = \left[\int_0^{x_\infty} Y_k^{-1}(p_j(k)\,,\,\xi)\; \partial_p A_k(p\,,\,\xi)\; |_{p=p_j(k)}\; Y_k(p_j(k)\,,\,\xi) d\xi\right]^{-1} \tag{34}$$

Because of the length of the expression (33), we introduce the auxiliary operator $M_k(p_j(k)\,,\,x) \in L(\mathscr{L}_k^2(C_1)\,,\,\mathscr{L}_k^2(C_1))$ by

$$M_k(p_j\,(k)\,,\,x) f_k(p_j(k) = \int_0^{x_\infty} Y_k(p_j\,(k)\,,\,x) \operatorname*{Res}_{p=p_j(k)} Y_k^{-1}(p_j\,(k)\,,\,\xi) f_k(p_j\,(k)\,,\,\xi)\, d\xi \tag{35}$$

Thus, instead of (33), we may write

$$y_k(x\,,\,t) = \sum e^{p_j(k)t} M_k(p_j(k)\,,\,x) f_k(p_j(k)) \tag{36}$$

*The effect of the group of motion of the Euclidean plane on (36)*

Because of the commutation relation

$$\check{T}_k(g) A_k = A_k \check{T}_k(g) \tag{37}$$

there is no interaction between the representation $\check{T}_k(g)$ , and $A_k$ . Consequently, $\check{T}_k(g) y_k$ is the solution of the boundary problem

$$d_x(\check{T}_k(g)\, y_k) = A_k \check{T}_k(g)\, y_k + \check{T}_k(g)\, f_k \tag{38}$$

and for this reason the use of (36) results in

$$\check{T}_k(g)y_k\,(x\,,\,t) \cong \sum e^{p_j(k)t}\, M_k(p_j(k)\,,\,x)\; \check{T}_k(g)f_k(p_j(k)) \tag{39}$$

The relation (39) shows how the projection $\check{T}_k(g)y_k$ of $\check{T}_g y$ behaves in the invariant subspace $\mathscr{L}^2_k(C_1)$ of $\mathscr{L}^2(\mathfrak{R}^2)$ . Thus, the use of

$$\check{T}_g y = 2\pi \int_{\mathfrak{R}^+} kdk\; \check{T}_k(g)y_k \tag{40}$$

(see (A18)) enables us to obtain the solution

$$\check{T}_g y = 2\pi \int_{\mathfrak{R}^+} kdk \sum_j e^{p_j(k)t}\, M_k(p_j(k)\,,\,x)\; \check{T}_k(g)f_k\,(p_j(k)) \tag{41}$$

in $\mathscr{L}^2(\mathfrak{R}^2)$ .

For the matrix elements of $\check{T}_g y$ we see that the expansion of (41) with respect to the basis (A19) supplies us with the series development

$$y(x\,,\,\hat{\mathbf{a}}\,,\,t) = 2\pi \int_{\mathfrak{R}^+} kdr \sum_j e^{p_j(k)t}\, h^n(0)\; M_k^{nl}\; \check{T}_\kappa^{lm}\; f_k^m \tag{42}$$

where a repeated index refers to the summation from $-\infty$ to $+\infty$ . Further, $M_k^{nl} = (h^n\,,\,M_k(p_j(k)\,,\,x)h^l)_{\mathscr{L}^2_k(C_1)}$ and $f_k^m = (h^m\,,f_k\,(p_j(k))_{\mathscr{L}^2_k(C_1)}$ . $\{h^n(0)\}$ is the basis (A19) at the value $\varphi = 0$ . (A24) shows in turn that the translational part (i.e. $\theta = 0$) of $\check{T}_n^{lm}$ is of the form

$$\check{T}_k^{lm} = i^{m-l}\, e^{i(m-l)\varphi}\; J_{m-l}(ka)$$

where $J_{m-l}(ka)$ , is the Bessel function of the order $m-l$ .

## *Conclusions*

The procedure above yielded first the modal expansion (36) valid in the invariant subspace $\mathscr{L}^2_k(C_1)$ . Afterwards the use of the interaction of the boundary value system with representation $\check{T}_g$ equipped us with the solution in (42), which is valid throughout the space $\mathscr{L}^2(\mathfrak{R}^2)$ .

The reader who is seeking guidelines for the numerical determination of $Y_k$ is referred to the study by SAASTAMOINEN [4], where some aspects concerning the numerical determination of $Y_k$ have been treated in the case of a spherically symmetric problem.

*Acknowledgements:* I wish to express my thanks to Prof. J. Segercrantz for his reading and to Miss A. Lipponen for her typewriting the manuscript.

Finally, my thanks are due to the Sohlberg Foundation of the Societas Scientiarum Fennica for financial support.

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## *Appendix*

### *The group of motion of the Euclidean plane*

The representation of the group of motion of the Euclidean plane $\mathfrak{R}^2$ are sought in the spaces $\mathcal{L}^2(\mathfrak{R}^2)$ and $\mathcal{L}^2_k(C_1)$, which are defined below.

$\mathcal{L}^2(\mathfrak{R}^2)$ consists of the complex, square-integrable and $\mathcal{L}^2(\mathfrak{R}^2) = \sum_i \oplus L^2(\mathfrak{R}^2)$ -valued functions on $\mathfrak{R}^2$. The summation over (now and later) goes from one to four in the case of radial modes and from one to two in the case of transverse modes. For the scalar product and the norm in $\mathcal{L}^2(\mathfrak{R}^2)$, we define

— The scalar product in $\mathcal{L}^2(\mathfrak{R}^2)$

$$(z\,,\,y)_{\mathcal{L}^2(\mathfrak{R}^2)} = \sum_i (z_i\,,\,y_i)_{L^2(\mathfrak{R}^2)} \tag{A1}$$

where $z$ and $y \in \mathcal{L}^2(\mathfrak{R}^2)$ and $z_i$ and $y_i \in L^2(\mathfrak{R}^2)$.

— The norm in $L^2(\mathfrak{R}^2)$

$$\|y\|^2_{\mathcal{L}^2(\Re^2)} = \sum_i \|(y_i)\|^2_{L^2(\Re^2)} \;, \tag{A2}$$

In (A1) and (A2) the terms $(z_i, y_i)_{L^2(\Re^2)}$ and $\|y_i\|_{L^2(\Re^2)}$ represent, respectively, the scalar product and the norm in the space $L^2(\Re^2)$ (complex, square-integrable functions on $\Re^2$) and are defined below
— The scalar product in $L^2(\Re^2)$

$$(z_i\,, y_i)_{L^2(\Re^2)} = \int_{\Re^2} z_i{}^*(\hat{\mathrm{x}}) y_i(\hat{\mathrm{x}}) d\hat{\mathrm{x}} \tag{A3}$$

— The norm in $L^2(\Re^2)$

$$\|y_i\|^2_{L^2(\Re^2)} = \int_{\Re^2} y_i{}^*(\hat{\mathrm{x}}) y_i(\hat{\mathrm{x}}) d\hat{\mathrm{x}} \tag{A4}$$

The space $\mathcal{L}^2_k(C_1)$ corresponds to the complex, square-integrable and $\mathcal{L}^2_k(C_1) = \sum_i \oplus L^2_k(C_1)$ -valued functions on the unit circle $(C_1)$ and is defined with the aid of the scalar product and the norm given below
— The scalar product in $\mathcal{L}^2_k(C_1)$

$$(z_k\,, y_k)_{\mathcal{L}_k^2(C_1)} = \sum_i (z_{ki}\,, y_{ki})_{L^2_k(C_1)} \tag{A5}$$

— The norm in $\mathcal{L}^2_k(C_1)$

$$\|y_k\|_{\mathcal{L}_k^2(C_1)} = \sum_i \|y_{ki}\|^2{}_{L^2_k(C_1)} \tag{A6}$$

The corresponding quantities in the component spaces $L^2_k(C_1)$ are de -fined in accordance with (A3) and (A4) as follows

$$(z_{ki}\,, y_{ki})_{L^2_k(C_1)} = \frac{1}{2\pi} \int_0^{2\pi} z^*_{ki}(\varphi) y_{ki}(\varphi) d\varphi \tag{A7}$$

— The norm in $L^2_k(C_1)$

$$\|y_{ki}\|_{L_k{}^2(C_1)} = \frac{1}{2\pi} \int_0^{2\pi} y^*_{ki}(\varphi)\, y_{ki}(\varphi) d\varphi \tag{A8}$$

For the treatment of the group of motion of the Euclidean plane, we follow in this paper the procedure adopted in the monograph by VILENKIN [5].

The group in question consists of the rotations and translations in the Euclidean plane $\mathfrak{R}^2$. In other words

$$g\hat{\mathbf{x}} = t_{\hat{a}} t_\theta \hat{\mathbf{x}} \tag{A9}$$

where $t_\theta$ rotates $\hat{\mathbf{x}}$ from $\mathfrak{R}^2$ by the amount of the polar angle $\theta$ and $t_{\hat{a}}$ correspondingly translates $\hat{\mathbf{x}}$ from $\mathfrak{R}^2$ by the amount of the vector $\hat{\mathbf{a}}$. I.e.

$$t_{\hat{a}}\hat{\mathbf{x}} = \hat{\mathbf{x}} + \hat{\mathbf{a}} \tag{A10}$$

We first define a representation $T_g$ in $\mathcal{L}^2(\mathfrak{R}^2)$ by

$$(T_g y)\,(\hat{\mathbf{x}}) = y(g^{-1}\hat{\mathbf{x}}) \tag{A11}$$

where $g^{-1}$ is the inverse of $g$. The present paper, however, is concerned with the Fourier-transformed version of $T_g$. Therefore introduction of the Fourier transformation

$$y(\hat{\mathbf{k}}) = \frac{1}{2\pi}\int\limits_{\mathfrak{R}^2} e^{i(\hat{k},\hat{x})} y(\hat{\mathbf{x}})d\hat{\mathbf{x}} \tag{A12}$$

(no confusion should be caused by the use of the same letter $y$ on both sides of the transformation) makes it possible to define the Fourier equivalent $\check{T}_g$ of $T_g$ by

$$(\check{T}_g y)\,(\check{\mathbf{k}}) = \frac{1}{2\pi}\int\limits_{\mathfrak{R}^2} e^{i(\hat{k},\hat{x})}\, T_g\, y(\hat{\mathbf{x}})d\hat{\mathbf{x}} \tag{A13}$$

Further, as a result of using (A9) together with the invariance of $d\hat{\mathbf{x}}$ with respect to $g$, we obtain instead of (A13), the more explicit form

$$(\check{T}_g y)\,(\hat{\mathbf{k}}) = e^{i(\hat{k},\hat{a})} y(t_{-\theta}\hat{\mathbf{k}}) \tag{A14}$$

(A14) shows in turn that $\check{T}_g$ induces the representation $\check{T}_k(g)$ in $\mathcal{L}_k^2(C_1)$ by

$$T_k(g) y_k(\hat{\mathbf{k}}^0) = e^{ik(\hat{k}^0,\hat{a})}\, y_k(t_{-\theta}\hat{\mathbf{k}}^0) \tag{A15}$$

where $\hat{\mathbf{k}}^0$ is the unit in the direction of $\hat{\mathbf{k}}$.

To see how $\check{T}_g$ and $\check{T}_k(g)$ are interrelated, we take arbitrary $y \in \mathcal{L}^2(\mathfrak{R}^2)$ and form decomposition

$$\|y\|_{\mathcal{L}^2(\Re^2)} = 2\pi \int\limits_{\Re^+} kdk \, \|y_k\|^2_{\mathcal{L}^2_k(C_1)} \tag{A16}$$

where $\Re^+$ is the positive real axis. As a consequence of

$$\mathcal{L}^2(\Re^2) = 2\pi \int\limits_{\Re^+} kdk \, \mathcal{L}^2_k(C_1) \tag{A17}$$

Finally, using the fact that $\check{T}_g \, y \in \mathcal{L}^2(\Re^2)$ , (A16) and (A17) also show

$$\check{T}_g \, y = 2\pi \int\limits_{\Re^+} kdk \, \check{T}_k(g) y_k \tag{A18}$$

For the matrix elements $\check{T}_k^{nm}$ of $\check{T}_k(g)$ , we introduce the basis

$$\{h^n\}_{n=-\infty}^{\infty} = \{e^{in\varphi}\}_{n=-\infty}^{\infty} \tag{A19}$$

into $\mathcal{L}^2_k(C_1)$ and express (A15) in the form

$$\check{T}_k(g) y_k(\varphi) = e^{i\,ka\cos(\varphi-\psi)} \, T_\theta y_k(\varphi) \tag{A20}$$

In (A20) $\varphi$ and $\psi$ are the polar angles of $\hat{\mathbf{k}}^0$ and $\hat{\mathbf{a}}$ , respectively. $T_\theta$ in turn is the representation of the rotation $t_\theta$ in $\mathcal{L}^2_k(C_1)$ . Finally, we define the matrix elements by

$$\check{T}_k^{nm} = (h^n \, , \check{T}_k(g) h^m)_{\mathcal{L}^2_k(C_1)} \tag{A21}$$

Accordingly, the substitution of (A20) into (A21), together with the use of the relation $T_\theta h^m = e^{im(\varphi-\theta)}$ , gives

$$\check{T}_k^{nm} = \frac{1}{2\pi} \int\limits_0^{2\pi} e^{i\,ka\cos(\varphi-\psi) + i(m-n)\varphi} \, e^{-im\theta} d\varphi \tag{A22}$$

From which after the change $\varphi = \frac{\pi}{2} + \psi - \alpha$ of the indepedent variable

$$\check{T}_k^{nm} = e^{i(m-n)\frac{\pi}{2}} e^{i(m-n)\psi} e^{-im\theta} \, \frac{1}{2\pi} \int\limits_0^{2\pi} e^{i\,ka\sin\alpha - i\,(m-n)\alpha} d\alpha \tag{A23}$$

Finally, using the definition

$$J_l(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix\sin\alpha - il\alpha} d\alpha$$

of the Bessel function $J_l(x)$ of order $l$, we obtain

$$\check{T}^{nm} = i^{m-n} e^{i(m-n)\psi} e^{-im\theta} J_{m-n}(ka) \tag{24}$$