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A SIMPLIFIED THEORY OF THE ANNUAL VARIATION OF THE GENERAL CIRCULATION*)

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Abstract

A two level quasi-geostrophic model incorporating a simple Newtonian form of diabatic heating and internal as well as boundary layer friction is used to study the annual variation of certain aspects of the general circulation. The model predicts the annual variation of the zonally averaged winds at the two levels as well as the zonal mean of the temperature. The momentum and heat transports by the large-scale eddies in the atmosphere are incorporated in the study through the use of exchange coefficients for the transports of heat and quasi-geostrophic potential vorticity. These exchange coefficients provide an indirect specification of the momentum transport by the eddies.

The investigation is limited to meridional variations which may be described by a single sinusoidal component, and to annual variations described by the annual mean and the first Fourier component of the annual variation. It is shown that the model is capable of predicting correctly the annual variation of the mean zonal wind, the eddy heat transport and certain aspects of the annual variation of the momentum transport and of atmospheric energetics. The main discrepancy between the computed results and atmospheric observations is that the predicted annual variation of all energy quantities is too large.

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1. *Introduction*

In a series of investigations [5], [6], [7], it has been attempted to construct atmospheric models which may be used to simulate the behavior of the atmosphere on an annual and seasonal basis. The limited purpose of these models has been to predict the behavior of the zonally averaged state of the atmosphere. It has been known for some time that there is a most important interaction between the zonally averaged quantities in the atmosphere and the deviation from this average, *i.e.* the eddies in the atmosphere. The changes of the zonally averaged winds are for example influenced directly by the convergence of the eddy momentum transport just as the changes in the zonally average temperature are governed to some extent by the convergence of the eddy transport of sensible heat in addition to other processes.

In the earlier investigations [5], [6], it has been assumed that one may describe the meridional transport of sensible heat by an empirical relation where the transport is related to the meridional temperature gradient through an exchange coefficient which in general is a function of latitude, pressure and time. Such a relation is possible in the troposphere because observational studies show that the transport of sensible heat in this part of the atmosphere is from the region of higher to the region of lower temperature. An analogous simple relation is not possible for the meridional transport of relative momentum by the eddies because it is observed that this transport at most places in the troposphere is from regions of low to regions of high relative momentum. However, based upon an idea by GREEN [1] it was shown by WIIN-NIELSEN and SELA [7] that the meridional transport of quasi-geostrophic potential vorticity may be described using an exchange coefficient which also in general will depend upon latitude, pressure and time. As shown in [7] it is then possible to obtain a parameterization of the convergence of the momentum transport by the eddies using the exchange coefficients for potential vorticity and sensible heat. The investigation contained furthermore an empirical determination of the exchange coefficients for various levels throughout the troposphere.

In the earlier attempts to simulate the annual and seasonal behavior of the zonally averaged quantities [5], [6], it was assumed that the effects of the momentum transport by the eddies on the zonally averaged quantities may be disregarded in comparison with the effects of the transport of sensible heat. While the neglect of the momentum transport is justified for the largest meridional scale it was also apparent from the investigations that the details of the predicted zonal wind profiles were seriously in error both with respect to the meridional and the vertical distributions. It was for example a consequence of the assumption that the predicted wind distribution at the ground (100 cb) was identically zero in the earlier model.

It is the purpose of this paper to describe the results of an investigation where the assumption concerning the neglect of the momentum transport has been removed and replaced by a parameterization of the convergence of the momentum transport as described in [7]. In this more general model it is now possible to simulate the energy conversion between the eddy kinetic and the zonal kinetic energy in addition to the processes already incorporated in the earlier papers.

The most general case in which the exchange coefficients vary with latitude will be reported elsewhere including the description of the numerical treatment of the problem which now most conveniently is handled as an initial value problem. In this case one can predict the zonal winds and the zonal temperatures as a function of latitude and time at selected levels by making a time-integration over several years. In this paper we shall restrict the solution to the very simple case in which we only consider a single component in the meridional direction. We are therefore mainly concerned with a prediction of the annual average and the annual variation of the largest scale in the meridional direction, and are leaving the questions concerning the detailed distribution to the later report.

Since our case can only be expected to describe certain general features of the annual variation of the general circulation we shall furthermore use Cartesian geometry and work on a betaplane, a procedure which further simplifies the mathematical treatment.

2. *Outline of the model*

The equations for the two level quasi-geostrophic model are

$$\frac{\partial Q_1}{\partial t} + \nabla \cdot (Q_1 \vec{v}_1) = \gamma q^2 (\psi_T - \psi_E) - 2 A \zeta_T \tag{2.1}$$

$$\frac{\partial Q_3}{\partial t} + \nabla \cdot (Q_3 \vec{v}_3) = - \gamma q^2 (\psi_T - \psi_E) + 2 A \zeta_T - \varepsilon (\zeta_* - 2 \zeta_T) \tag{2.2}$$

where

$$Q_1 = f + \zeta_1 - q^2\psi_T \tag{2.3}$$

$$Q_3 = f + \zeta_3 + q^2\psi_T \tag{2.4}$$

The reader is referred to [6] for the explanation of the remaining symbols in (2.1) and (2.2). Forming zonal averages of (2.1) and (2.2) and using a Cartesian geometry we have:

$$\frac{\partial Q_{1z}}{\partial t} + \frac{\partial (Q_1 v_1)_z}{\partial y} = \gamma q^2 (\psi_{Tz} - \psi_{Ez}) - 2\,A\zeta_{Tz} \tag{2.5}$$

$$\frac{\partial Q_{3z}}{\partial t} + \frac{\partial (Q_3 v_3)_z}{\partial y} = -\gamma q^2 (\psi_{Tz} - \psi_{Ez}) + 2\,A\zeta_{Tz} - \varepsilon\,(\zeta_* - 2\zeta_r) \tag{2.6}$$

Based upon the results obtained in [7] we make the approximations that

$$(Q_1 v_1)_z = -K_1 \frac{\partial Q_{1z}}{\partial y} \tag{2.7}$$

$$(Q_3 v_3)_z = -K_3 \frac{\partial Q_{3z}}{\partial y} \tag{2.8}$$

and (2.5) and (2.6) become, assuming here for simplicity that K_1 and K_3 are not functions of y:

$$\frac{\partial Q_{1z}}{\partial t} = K_1 \frac{\partial^2 Q_{1z}}{\partial y^2} + \gamma q^2 (\psi_{Tz} - \psi_{TE}) - 2\,A\zeta_{TZ} \tag{2.9}$$

$$\frac{\partial Q_{3z}}{\partial t} = K_3 \frac{\partial^2 Q_{3z}}{\partial y^2} - \gamma q^2 (\psi_{Tz} - \psi_{TE}) + 2\,A\zeta_{Tz} - \varepsilon\,(\zeta_{*z} - 2\,\zeta_{Tz}) \tag{2.10}$$

We note from (2.3) and (2.4) that

$$Q_{1z} = f + \frac{\partial^2 \psi_{1z}}{\partial y^2} - q^2\psi_{Tz} \tag{2.11}$$

$$Q_{3z} = f + \frac{\partial^2 \psi_{3z}}{\partial y^2} + q^2\psi_{Tz} \tag{2.12}$$

Denoting in general

$$(\quad)_* = \tfrac{1}{2}\,[(\quad)_1 + (\quad)_3] \tag{2.13}$$

$$(\quad)_T = \tfrac{1}{2}\,[(\quad)_1 - (\quad)_3] \tag{2.14}$$

we find that

$$Q_{*z} = f + \frac{\partial^2 \psi_{*z}}{\partial y^2} \tag{2.15}$$

$$Q_{Tz} = \frac{\partial^2 \psi_{Tz}}{\partial y^2} - q^2 \psi_{Tz} \tag{2.16}$$

Adding and subtracting (2.9) and (2.10) and using in addition

$$K_* = \tfrac{1}{2}(K_1 + K_3) \tag{2.17}$$

$$K_T = \tfrac{1}{2}(K_1 - K_3) \tag{2.18}$$

we find

$$\begin{aligned} \frac{\partial}{\partial t}\left[\frac{\partial^2 \psi_{*z}}{\partial y^2}\right] = K_* \frac{\partial^4 \psi_{*z}}{\partial y^4} + K_T \left[\frac{(\partial^4 \psi_{Tz}}{\partial y^2} - q^2 \frac{\partial^2 \psi_{Tz}}{\partial y^2}\right] - \\ \frac{\varepsilon}{2}\left[\frac{\partial^2 \psi *_z}{\partial y^2} - 2\frac{\partial^2 \psi_{Tz}}{\partial y^2}\right] \end{aligned} \tag{2.19}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left[\frac{\partial^2 \psi_{Tz}}{\partial y^2} - q^2 \psi_{Tz}\right] = K_* \left[\frac{\partial^4 \psi_{Tz}}{\partial y^4} - q^2 \frac{\partial^2 \psi_{Tz}}{\partial y^2}\right] + K_T \frac{\partial^4 \psi_{*z}}{\partial y^4} \\ + \gamma q^2 (\psi_{Tz} - \psi_{Ez}) - 2A \frac{\partial^2 \psi_{Tz}}{\partial y^2} + \frac{\varepsilon}{2}\left[\frac{\partial^2 \psi_{*z}}{\partial y^2} - 2\frac{\partial^2 \psi_{Tz}}{\partial y^2}\right] \end{aligned} \tag{2.20}$$

Equations (2.19) and (2.20) have ψ_{*z} and ψ_{Tz} as the only dependent variables. The equations can in general be integrated. As mentioned earlier we shall here in the initial treatment restrict ourselves to the case where

$$\psi_{*z} = \Psi_*(t) \cos \mu y\,,\ \psi_{Tz} = \Psi_T(t) \cos \mu y\,,\ \psi_{Ez} = \Psi_E(t) \cos \mu y \tag{2.21}$$

where $\mu = \pi/D$, and D is the width of the channel. Substituting (2.21) in (2.19) and (2.20) we get

$$\frac{d\Psi_*}{dt} = -\left(\mu^2 K_* + \frac{\varepsilon}{2}\right)\Psi_* - (\mu^2 K_T + q^2 K_T - \varepsilon)\Psi_T \tag{2.22}$$

$$\begin{aligned} \left(1 + \frac{q^2}{\mu^2}\right)\frac{d\Psi_T}{dt} = -\left(\mu^2 K_T - \frac{\varepsilon}{2}\right)\Psi_* - \\ \left(\mu^2 K_* + q^2 K_* + \varepsilon + 2A + \gamma \frac{q^2}{\mu^2}\right)\Psi_T + \gamma \frac{q^2}{\mu^2}\Psi_E \end{aligned} \tag{2.23}$$

A particular solution of these equations are now obtained by writing the dependent variables in the form:

$$\begin{aligned} \Psi_*(t) &= \Psi_*^{(0)} + A_* \cos st + B_* \sin st \\ \Psi_T(t) &= \Psi_T^{(0)} + A_T \cos st + B_T \sin st \end{aligned} \tag{2.24}$$

while the forcing function $\Psi_E(t)$ has the form

$$\Psi_E(t) = \Psi_E^{(0)} + A_E \cos st \tag{2.25}$$

The implication of the form (2.25) is that we count the time from the maximum in $\Psi_E(t)$. (2.24) and (2.25) are substituted in (2.22) and (2.23). Equating time-independent values, terms which depend upon cos st, and terms which depend on sin st, respectively, we get the following equations:

$$\left(\mu^2 K_* + \frac{\varepsilon}{2}\right) \Psi_*^{(0)} + (\mu^2 K_T + q^2 K_T - \varepsilon)\, \Psi_T^{(0)} = 0 \tag{2.26}$$

$$\left(\mu^2 K_T - \frac{\varepsilon}{2}\right) \Psi_*^{(0)} + \left(\mu^2 K_* + q^2 K_* + \varepsilon + 2A + \gamma \frac{q^2}{\mu^2}\right) \Psi_T^{(0)} = \gamma \frac{q^2}{\mu^2} \Psi_T^{(0)}$$

and

$$\begin{aligned}
&- sA^* + \left(\mu^2 K_* + \frac{\varepsilon}{2}\right) B_* + 0 \cdot A_T + \{(\mu^2 + q^2) K_T - \varepsilon\}\, B_T = 0 \\
&\left(\mu^2 K_* + \frac{\varepsilon}{2}\right) A_* + s \cdot B_* + \{(\mu^2 + q^2) K_T - \varepsilon\}\, A_T + 0 \cdot B_T = 0 \\
&0 \cdot A_* + \left(\mu^2 K_T - \frac{\varepsilon}{2}\right) B_* - \left(1 + \frac{q^2}{\mu^2}\right) s \cdot A_T + \left\{(\mu^2 + q^2) K_* + \varepsilon + 2A + \gamma \frac{q^2}{\mu^2}\right\} B_T = 0 \\
&\left(\mu^2 K_T - \frac{\varepsilon}{2}\right) A_* + 0 \cdot B_* + \left\{(\mu^2 + q^2) K_* + \varepsilon + 2A + \gamma \frac{q^2}{\mu^2}\right\} A_T + \left(1 + \frac{q^2}{\mu^2}\right) s \cdot B_T = \gamma \frac{q^2}{\mu^2} A_E
\end{aligned} \tag{2.27}$$

The solutions of the systems (2.26) and (2.27) constitute the complete solution of the problem. In order to obtain the solution we must select the numerical values of all parameters.

3. *Solution of the problem*

Most of the numerical values for the various constants can be adopted from the earlier studies [5], [6]. We list these values as follows without further justification:

$$q^2 = \frac{2f_0^2}{\sigma P^2} = 4 \times 10^{-12}\,\mathrm{m}^{-2}$$

$$\mu^2 = \frac{\pi^2}{D^2} = 0.1 \times 10^{-12}\,\mathrm{m}^{-2}, \quad \text{corresponding to} \quad D = 10^7\,\mathrm{m}$$

$$\varepsilon = 3.0 \times 10^{-6}\,\mathrm{sec}^{-1}$$

$$s = 0.2 \times 10^{-6}\,\mathrm{sec}^{-1}$$

$$A = 0.6 \times 10^{-6}\,\mathrm{sec}^{-1}.$$

The constant values of K_1 and K_3 were obtained from the annual mean values of the exchange coefficient for the levels 30 cb and 70 cb by computing the meridional average. In this way one finds that

$$K_1 = 0.9 \times 10^6\,\mathrm{m}^2\,\mathrm{sec}^{-1}$$

$$K_3 = 2.0 \times 10^6\,\mathrm{m}^2\,\mathrm{sec}^{-1}$$

and therefore

$$K_* = 1.4 \times 10^6\,\mathrm{m}^2\,\mathrm{sec}^{-1}$$

$$K_T = -\,0.6 \times 10^6\,\mathrm{m}^2\,\mathrm{sec}^{-1}.$$

The numerical values of $\Psi_E^{(0)}$ and A_E were taken from the calculation given in [5]. They are:

$$\Psi_E^{(0)} = 24.8 \times 10^6\,\mathrm{m}^2\,\mathrm{sec}^{-1}$$

and

$$A_E = 17.4 \times 10^6\,\mathrm{m}^2\,\mathrm{sec}^{-1}.$$

Using the values listed above it is straightforward to solve the systems (2.26) and (2.27). One obtains the following values of the six unknowns in (2.26) and (2.27):

$$\Psi_*^{(0)} = 60.5 \times 10^6\,\mathrm{m}^2\,\mathrm{sec}^{-1}$$

$$\Psi_T^{(0)} = 18.8 \times 10^6\,\mathrm{m}^2\,\mathrm{sec}^{-1}$$

$$A_* = 33.7 \times 10^6\,\mathrm{m}^2\,\mathrm{sec}^{-1}$$

$$B_* = 19.1 \times 10^6 \text{ m}^2 \text{ sec}^{-1}$$

$$A_T = 11.2 \times 10^6 \text{ m}^2 \text{ sec}^{-1}$$

$$B_T = 4.7 \times 10^6 \text{ m}^2 \text{ sec}^{-1}$$

which is the solution of the simple problem stated here. It is probably most instructive to express the solution in terms of the zonal wind components at the various levels. In calculating the zonal winds we use the formula that

$$U = -\frac{\partial \psi_z}{\partial y} = \mu \Psi(t) \sin \mu y = U(t) \sin \mu y \tag{3.1}$$

where

$$U(t) = \mu \Psi(t) \tag{3.2}$$

Using the values given above we find after some manipulations

$$\begin{aligned} U_1 &= 24.9 + 16.0 \cos s\,(t - 28) \\ U_3 &= 13.1 + 8.4 \cos s\,(t - 32) \end{aligned} \tag{3.3}$$

where the annual mean values and the amplitude of the first harmonic are given in m sec^{-1}, while the phase is given in the unit: days.

The wind distribution at 100 cb, denoted by a subscript 4, is calculated from the extrapolation formula

$$U_4 = \frac{3}{2} U_3 - \frac{1}{2} U_1 \tag{3.4}$$

and we obtain

$$U_4 = 7.2 + 4.7 \cos s\,(t - 41)\,.$$

We may consider the quantities $U(t)$ as a measure of the maximum winds at the different levels. The prediction of this model is therefore that the maximum winds at the upper level (25 cb) occur 28 days after the maximum in the forcing function $\Psi_E(t)$, which in turn is a measure of the temperature difference between equator and pole in the equilibrium temperature. The model also predicts that the windmaximum occurs 32 and 41 days after the same time at the levels 3 (75 cb) and 4 (100 cb), respectively. A comparison of these predictions by the model with the observed behavior of the atmosphere will be made in the next section.

Several quantities of considerable dynamical significance can be computed from the solutions which have been obtained. We shall first compute the vertical velocity, which can be obtained from the thermodynamic equation. Writing the thermodynamic equation in the form:

$$\frac{\partial \psi_T}{\partial t} + \vec{v} \cdot \nabla \psi_T - \frac{\sigma P}{2 f_0} \omega = \frac{1}{2 f_0} \frac{R}{c_P} H \tag{3.6}$$

as in any two level quasi-geostrophic model we obtain by taking the zonal average that

$$\omega_z = \frac{2 f_0}{\sigma P} \left[\frac{\partial \psi_T}{\partial t} + \frac{\partial (\psi_T v)_z}{\partial y} \right] - \frac{R}{c_P} \frac{1}{\sigma P} H_z \tag{3.7}$$

In (3.7) we introduce the expression for H_z and in addition

$$(\psi_T v)_z = - K_H \frac{\partial \psi_{Tz}}{\partial y} \tag{3.8}$$

as used in (6) and find

$$\omega_z = \frac{P}{f_0} q^2 \left[\frac{\partial \psi_{Tz}}{\partial t} - K_H \zeta_{Tz} + \gamma (\psi_{Tz} - \psi_{TE}) \right] \tag{3.9}$$

The expressions (2.21) are substituted in (3.9), and we find

$$\omega_z = \Omega (t) \cos \mu y \tag{3.10}$$

where

$$\Omega (t) = \frac{P}{f_0} q^2 \left[\frac{d \Psi_T}{dt} + \mu^2 K_H \Psi_T + \gamma (\Psi_T - \Psi_E) \right] \tag{3.11}$$

The values of $\Omega (t)$ can be computed immediately using the solution given at the beginning of this section. We find:

$$\Omega (t) = 10^{-6} (1.76 + 1.22 \cos s (t - 46) \tag{3.12}$$

where it has been assumed that K_H in (3.8) is equal to 1.7×10^6 m sec^{-1}, a value obtained as a meridional average from the data given in [7].

The single meridional component used in this solution does not permit more than a single meridional cell in the mean meridional circulation. However, (3.12) shows together with (3.10) that the single cell is an indirect cell with sinking motion in the low latitudes and rising motion in the high latitudes during the whole year. The intensity of this cell is strongest in February, where the maximum sinking motion is 2.98×10^{-6} cb $\sec^{-1} \simeq -0.4$ mm $\sec^{-1}$, and of the smallest intensity in August. The direction of mean meridional circulation is opposite to the direct cell created by the diabatic heating. This cell can be calculated from (3.9) by finding the contribution from the last term. Using the same technique as before we find that the mean meridional circulation due to the heating can be described by the expression

$$\Omega_H(t) = \frac{P}{f_0} q^2 \gamma \, (\Psi_T - \Psi_E) \tag{3.13}$$

or

$$\Omega_H(t) = 10^{-6} \{-4.76 + 6.19 \cos s\,(t - 143)\} \tag{3.14}$$

When (3.14) is converted to the vertical velocity, measured in mm $\sec^{-1}$, using the formula $\omega = -g\varrho w$, we find that

$$W_H(t) = 0.95 + 1.24 \cos s\,(t + 37) \tag{3.15}$$

(3.15) shows that the intensity of the mean meridional circulation due to the heating is at its maximum in late November. The circulation is of a direct nature except during a period in early summer where the direction of the cell is reversed. The change in direction is related to the difference between the equilibrium temperature and the actual temperature in the model. This can be seen by computing the difference

$$\Psi_E - \Psi_T = 10^6 \times \{5.95 + 7.74 \cos s\,(t + 37)\} \tag{3.16}$$

which shows that $\Psi_E - \Psi_T$ is negative during the early summer.

We shall next calculate the transport of sensible heat predicted by the model. Using the exchange coefficient we get

$$(Tv)_z = -K_H \frac{\partial T_z}{\partial y} \tag{3.17}$$

The thermal wind relation is introduced in (3.17), and we find that

$$(Tv)_z = K_H \frac{f_0}{R} (u_{1z} - u_{3z}) \tag{3.18}$$

Using the values given in (3.3) we find that

$$(Tv)_z = \{7.10 + 4.58 \cos s\,(t - 23)\} \sin \mu y \tag{3.19}$$

showing that the model predicts a heat transport with a maximum in the middle of the channel and with a maximum heat transport in late January and a minimum, but still positive, transport in late July.

The convergence of the momentum transport can also be calculated from the solution. Using the parameterization given in [7] we find using the notation $M_1 = (u_1 v_1)_z$ that

$$- \frac{\partial M_1}{\partial y} = S_1 - \tilde{S}_1 \tag{3.20}$$

where

$$\tilde{S}_1 = \frac{1}{D} \int_0^D S_1 \, dy \tag{3.21}$$

and

$$S_1 = - \beta K_1 - \mu^2 K_1 u_{1z} + q^2 (K_H - K_1) u_{Tz} \tag{3.22}$$

Substituting the expressions for u_{1z} and u_{3z} and using the already adopted values for the exchange coefficients we find that

$$- \frac{\partial M_1}{\partial y} = 10^{-6} \{18{\cdot}12 + 11.72 \cos s\,(t - 22)\} \left\{\sin \mu y - \frac{2}{\pi}\right\} \tag{3.23}$$

The expression for the convergence of the momentum transport at level 3 is derived in an analogous manner, and the result is with $M_3 = (u_3 v_3)_z$

$$- \frac{\partial M_3}{\partial y} = 10^{-6} \{2.83 + 1.89 \cos s\,(t - 13)\} \left\{\sin \mu y - \frac{2}{\pi}\right\} \tag{3.24}$$

The expressions (3.23) and (3.24) may be integrated with respect to y in order to obtain the momentum transports. We find

$$M_1 = \{57.3 + 36.8 \cos s\,(t - 22)\} \{\cos \pi y_* - (1 - 2\, y_*)\} \tag{3.25}$$

$$M_3 = \{8.9 + 5.9 \cos s\,(t - 13)\} \{\cos \pi y_* - (1 - 2\, y_*)\} \tag{3.26}$$

We find that the model predicts a momentum transport which at each level is positive in the southern part of the channel and negative in the northern part of the channel in qualitative agreement with the observed momentum transport in the atmosphere. In addition, the momentum transport at the upper level is considerably larger than at the lower level, and the maximum momentum transport is in the later part of January at the upper level, but in the middle of January at the lower level. We note furthermore that the momentum transport at each level is qualitatively the same throughout the year.

The last quantities which will be computed from the solution of the model equations are all related to the energetics of the model. It is possible to compute the amounts of available potential and kinetic energy, the generation of zonal available potential energy, the energy conversions from zonal available potential to eddy available potential energy, from zonal available potential to zonal kinetic energy, and from eddy kinetic to zonal kinetic energy, and finally the dissipation of zonal kinetic energy.

Some of the integrals are of the type

$$\frac{1}{LD}\int_0^L\int_0^D X(t)\,Y(t)\cos^2\mu y\,dx\,dy = \tfrac{1}{2}X(t)\,Y(t) \tag{3.27}$$

In other integrals $\cos\mu y$ in (3.27) is replaced by $\sin\mu y$, but it is seen that the result is unchanged. The only other integral is $C(K_E, K_z)$ which is of the type

$$\frac{1}{LD}\int_0^L\int_0^D X(t)\,Y(t)\sin\mu y\left(\sin\mu y-\frac{2}{\pi}\right)dx\,dy \tag{3.28}$$

A direct evaluation of (3.28) gives she result

$$\left(\tfrac{1}{2}-\frac{4}{\pi^2}\right)X(t)\,Y(t) = 0.1\,X(t)\,Y(t) \tag{3.29}$$

The formulas for various integrals are listed below:

$$A_z = \frac{P}{g}\,q^2\,\frac{1}{D}\int_0^D \psi_{Tz}^2\,dy$$

$$K_z = \frac{P}{g}\,\frac{1}{D}\int_0^D \tfrac{1}{2}\left(u_{1z}^2+u_{3z}^2\right)dy$$

$$G(A_z) = 2\,\frac{P}{g}\,q^2\,\gamma\,\frac{1}{D}\int_0^D \psi_{Tz}\left(\psi_{Ez}-\psi_{Tz}\right)dy$$

$$C(A_z, A_E) = 2\frac{P}{g} q^2 K_H \frac{1}{D}\int_0^D u_{T_z}^2 \, dy \tag{3.30}$$

$$C(A_z, K_z) = -\frac{2f_0}{g}\frac{1}{D}\int_0^D \psi_{T_z}\omega_z \, dy$$

$$C(K_E, K_z) = -\frac{P}{g}\frac{1}{D}\int_0^D \left(u_{1z}\frac{\partial M_1}{\partial y} + u_{3z}\frac{\partial M_3}{\partial y}\right) dy$$

$$D(K_z) = \frac{P}{g}\frac{1}{D}\int_0^D (\varepsilon \cdot u_{3z}u_{4z} + \tfrac{1}{2}A(u_{1z} - u_{3z})^2 \, dy$$

It is seen that the simplicity of the model creates some arbitrariness in the evaluation of some of these integrals. As seen from the model equations (2.9) and (2.10) we need only the exchange coefficients K_1 and K_3 to find the solution, while K_H is unnecessary for this purpose. K_H is, however, needed for the evaluation of ω_z and the momentum transports M_1 and M_3. In terms of the energy conversions we need the value of K_H directly or indirectly in $C(A_z, A_E)$, $C(A_z, K_z)$ and $C(K_E, K_z)$. Because of this arbitrariness there is no unique energy diagram corresponding to the model, because a different value of K_H will give a variation in the energy diagram. In addition, we can not expect under these circumstances to obtain a balanced energy diagram in the sense that each energy reservoir will have as much inflow as outflow of energy per unit time. Since it is desirable to have the last requirement fulfilled, we have proceeded in the following way. $D(K_z)$ and $C(K_E, K_z)$ were computed first according to the formulas given in (3.30). $C(A_z, K_z)$ was then determined from the relation

$$\frac{dK_z}{dt} = C(A_z, K_z) + C(K_E, K_z) - D(K_z) \tag{3.31}$$

$C(A_z, A_E)$ was next computed from (3.30) and $G(A_z)$ was calculated from the relation

$$\frac{dA_z}{dt} = G(A_z) - C(A_z, A_E) - C(A_z, K_z) \tag{3.32}$$

It should be pointed out that several other procedures are possible.

The results of this calculation are:

$$\begin{aligned} A_z &= 4272 + 4567 \cos s\,(t - 23) \\ K_z &= 1193 + 1267 \cos s\,(t - 29) \\ G\,(A_z) &= 2.14 + 2.46 \cos s\,(t - 1) \\ C\,(A_z\,, A_E) &= 1.46 + 1.56 \cos s\,(t - 24) \\ C\,(A_z\,, K_z) &= 0.68 + 0.72 \cos s\,(t - 21) \\ C\,(K_E\,, K_z) &= 0.30 + 0.32 \cos s\,(t - 25) \\ D\,(K_z) &= 0.98 + 1.05 \cos s\,(t - 35) \end{aligned} \qquad (3.33)$$

In (3.33) A_z and K_z are given in the unit kj m^{-2}, while the rest of the quantities are given in Watts m^{-2}. Several comments should be made regarding the results stated in (3.33). The arbitrariness of the calculation shows up clearly in the value of $C\,(A_z\,, K_z)$ which is positive in (3.33) in spite of the indirect mean meridional circulation. In addition, it is noted that the amplitude of the first Fourier component of the annual variation in all cases is larger than the annual mean value. This is clearly in error in the quantities A_z and K_z which by definition are positive throughout the year. The reason is naturally that we have included only the first Fourier component and therefore deal with a strongly truncated system. An example will illustrate this point. Consider for example

$$A_z = \tfrac{1}{2}\,\frac{P}{g}\,q^2\,\Psi_T^2 = \tfrac{1}{2}\,\frac{P}{g}\,q^2\,(\Psi_T^{(0)} + A_T \cos st + B_T \sin st)^2 \qquad (3.34)$$

When we include all terms in the evaluation of A_z we get

$$A_z = 4272 + 4567 \cos s\,(t - 23) + 1034 \cos 2\,st \qquad (3.35)$$

and this quantity is positive definite according to (3.34). A similar remark holds for all other quantities.

A comparison will be made between the results in (3.33) and observational studies in the next section but we mention here that the model predicts a time lag of 6 days between the maxima in A_z and K_z, that there is a time lag of 34 days between the maximum in $G\,(A_z)$ and $D\,(K_z)$, but only 20 days as the time lag between $G\,(A_z)$ and $C\,(A_z, K_z)$.

4. *Comparisons with observational studies.*

The comparison of the results obtained in section 3 with results obtained from an analysis of atmospheric data is difficult, because the model assumes a very simple meridional distribution. However, a qualitative comparison can be made.

The quantities U_1, U_3 and U_4 given in (3.3) and (3.5) are the maximum zonal winds at the different levels in the model. In order to compare the quantities with observed quantities it is therefore natural to make an analysis of the mean zonal winds close to the maximum winds in the atmosphere. Zonal winds were available for the year 1963, and the annual average and the first Fourier component of the zonal winds were computed at 32.5°N close to the maximum in the meridional direction. The results of this analysis were

$$\begin{aligned} u_z &= 26.5 + 18.0 \cos s\,(t - 30); \quad p = 20 \text{ cb} \\ u_z &= 22.2 + 15.1 \cos s\,(t - 30); \quad p = 30 \text{ cb} \\ u_z &= 12.8 + 10.0 \cos s\,(t - 32); \quad p = 50 \text{ cb} \\ u_z &= 6.5 + 5.8 \cos s\,(t - 34); \quad p = 70 \text{ cb} \\ u_z &= 3.0 + 3.2 \cos s\,(t - 40); \quad p = 85 \text{ cb} \end{aligned} \tag{4.1}$$

A comparison between (4.1) and (3.3) shows that the predicted winds are in reasonable agreement with the observed winds. Note, in particular that the predicted increase of the phase with increasing pressure is confirmed by the results listed in (4.1). In making the comparison between (3.3), (3.5) and (4.1) one should note that the results listed in (3.3), (3.5) are counted from the maximum in Ψ_E, while those in (4.1) are counted from January 1. However, as we shall see later, these points in time are only separated by a few days.

We have no detailed observational studies which can be used to make a comparison with the predicted mean meridional circulation, but earlier studies indicate at least that the maximum intensity is found in winter as indicated by (3.12).

Data are available for the transport of sensible heat permitting a comparison with (3.19). Again, we use data from the year 1963. Since the heat transport given in (3.19) is the maximum heat transport in the model it is natural to compare with data selected from a latitude close to the one displaying maximum heat transport. The latitude 45°N was selected, and an analysis of the heat transport in the layer from 50 cb to 30 cb was made. When the result is converted into the quantity $(Tv)_z$ we obtain

$$(Tv)_z = 7.94 + 6.29 \cos s\,(t - 23) \tag{4.2}$$

which shows that the heat transport is well predicted by the model.

In order to compare the computed momentum transports given in (3.25) and (3.26) with observational studies we have in a similar way made an analysis of momentum transport data from the year 1963 in this case at 40°N which is close to the maximum northward transport in the annual mean. The results at the various levels are listed below:

$$\begin{aligned} M &= 57.1 + 29.9 \cos s\,(t + 5), && 20\text{cb} \\ M &= 48.6 + 34.1 \cos s\,(t - 2), && 30\text{ cb} \\ M &= 22.2 + 14.8 \cos s\,(t - 15), && 50\text{ cb} \\ M &= 10.2 + 4.3 \cos s\,(t - 57), && 70\text{ cb} \\ M &= 7.7 + 3.2 \cos s\,(t - 145), && 85\text{ cb} \end{aligned} \tag{4.3}$$

showing that while the order of magnitude of both the annual average and the amplitude of the first Fourier component are correct we have an incorrect variation of the phase with pressure in the model as compared to the observational studies. It can be shown that this discrepancy can not be corrected by another choice of K_H for the given values of K_1 and K_3 used to produce the solution, if we want to maintain the fact that the momentum transport is larger at the upper level than at the lower level.

The results listed in (3.33) for the energetics of the model may be compared with those given by WIIN-NIELSEN [4] based on observations. It is seen that the annual mean values are of the correct order of magnitude. The same holds for the computed phase angles, while the amplitudes of the first Fourier component of the annual variation are too large in all cases. This is in all likelihood due to the simplicity of the model, and one may point to several factors which may correct the discrepancy. The most likely factor is the simplicity of the Newtonian form of diabatic heating which does not model the heating of the atmosphere in more than a qualitatively correct way. Other factors are a possible annual variation in the various parameters in the model such as ε, A, K_1, K_3 and K_H. However, we do not know too much about these variations at the present time from observational studies, and new knowledge is necessary to incorporate such variations in the theoretical calculations. On the other hand test calculations using the present model, but incorporating hypothetical annual variations of the parameters mentioned above, leads to the tentative conclusion that the main factor responsible for the discrepancy is the simplified diabatic heating.

5. *Variation of parameters*

In the solution presented in section 3 of this paper we have selected values of the parameters which are reasonable based on available evidence. However, it is realized that there is considerable uncertainty in each of the parameters which describe the model. It is the purpose of this section to discuss the results of a few calculations which will illustrate the sensitivity of the solution to the choice of the parameters.

We shall first select the extreme case in which we make the exchange coefficients equal to zero, *i.e.* $K_1 = K_3 = 0$. This means that the eddies in the atmosphere has no influence at all on the zonal flow. When the systems (2.26) and (2.27) are solved under this assumption we get the result:

$$\begin{aligned} U_1 &= 21.7 + 13.5 \cos s\,(t - 32) \\ U_3 &= 7.2 + 4.5 \cos s\,(t - 42) \\ U_4 &= 0.0 + 1.2 \cos s\,(t + 56) \end{aligned} \tag{5.1}$$

which shows that the interaction between the eddies and the zonal flow is of great importance to explain the annual variation. If this interaction is not present in the model we get no mean annual flow at 100 cb. We obtain in addition a too large phase difference between the zonal winds at 25 cb and 75 cb, and a completely erroneous phase angle at 100 cb. The same statement can be made for the solution presented by WIIN-NIELSEN [6] where the effects of the momentum transport was neglected compared to the heat transport. In that case there is not a mean annual flow at 100 cb either. The results presented in (3.3), (3,5) as compared with (5.1) and the solution presented in [6] show the importance of modeling the momentum transport in a correct way.

In the following series of experiments it was decided to keep the following parameters constant: $K_1 = 0.9 \times 10^6\ \mathrm{m^2\,sec^{-1}}$, $K_3 = 2.0 \times 10^6\ \mathrm{m^2\,sec^{-1}}$, $q^2 = 4 \times 10^{-12}\ \mathrm{m^{-2}}$ and $\mu^2 = 0.1 \times 10^{-12}\ \mathrm{m^{-2}}$, while the values of ε, A and γ were varied from experiment to experiment. As the basic solution we shall take the case described in details in section 3. This solution will be called experiment 1. The numerical values of the parameters in the other experiments are listed in Table 1.

Table 1

	γ, 10^{-6} sec^{-1}	ε, 10^{-6} sec^{-1}	A, 10^{-6} sec^{-1}
Exp. 1	0.4	3.0	0.6
Exp. 2	0.8	6.0	1.2
Exp. 3	0.2	1.5	0.3
Exp. 4	0.8	1.5	0.3
Exp. 5	0.2	6.0	1.2

The solutions for U_1, U_3 and U_4 in experiments 1—5 are given in Table 2, where we have listed the annual mean value $\bar{U}$, the amplitude of the first Fourier component U_A and the phase angle δ, measured in days for each experiment.

Table 2

	U_1			U_3			U_4		
	$\bar{U}_1$	U_{A1}	δ_1	$\bar{U}_3$	U_{A3}	δ_3	$\bar{U}_4$	U_{A4}	δ_4
Exp. 1	24.9	16.0	28	13.1	8.4	32	7.2	4.7	41
Exp. 2	23.6	16.1	15	10.6	7.2	19	4.1	2.8	28
Exp. 3	25.6	14.2	45	15.7	8.8	51	10.8	6.1	59
Exp. 4	35.3	23.4	24	21.7	14.4	30	14.9	10.1	38
Exp. 5	17.2	10.0	36	7.7	4.5	39	3.0	1.8	48

The results of experiment 2, where the intensity of the Newtonian heating and the dissipation both have been increased by a factor 2, show that the maxima of the first Fourier component has been displaced to an earlier time of the year by 13 days, but that the displacement is the same for both U_1, U_3 and U_4. In addition, we find some decrease in the annual mean value and the amplitude of the first component at levels 3 and 4, which actually brings the result of experiment 2 in somewhat closer agreement with the observed results listed in (4.1) in this regard.

Experiment 3 which is characterized by a low intensity of both heating and dissipation has changes which are opposite to those of experiment 2 as could be expected. However, the relative position of the first Fourier component at levels 1, 3 and 4 is the same in this experiment as in experiments 1 and 2, but the amplitudes at levels 3 and 4 is definitely too large as compared with the observed results in (4.1). A similar observation can be made for experiment 4 characterized by large heating and small dissipation, while experiment 5 gives results where both the annual mean values and the amplitudes of the first component are too small as compared to (4.1) at the upper level.

In summary, the results of this series of experiments are that the relative position of the first cmponent of the annual variation is about the same even for rather radical changes in the parameters determining the solution, and that the changes in the annual mean values and amplitudes are much smaller than the corresponding changes in the parameters. The main character of the solution is maintained even during a rather wide variation of the parameters.

6. *Summary and concluding remarks.*

A simple two level quasi-geostrophic model has been used to simulate the annual behavior of the atmosphere. The model incorporates diabatic heating in the form of Newtonian heating, internal and boundary layer friction and a parameterization of momentum and heat transport. The parameterization of the heat transport is made through the use of an exchange coefficient, while the parameterization of the momentum transport is obtained indirectly by defining an exchange coefficient for quasi-geostrophic potential vorticity. The combined use of the exchange coefficients for heat and potential vorticity leads to a relation between the convergence of the momentum transport and the mean zonal winds in the atmosphere.

The use of exchange coefficients represents an empirical and simple way of modeling the interaction between the zonal average and the large-scale eddies in the atmosphere. This approximation accomplishes the task of closing the system of equations describing the behavior of the zonally averaged quantities which therefore may be predicted without a detailed knowledge of the individual eddies. Another way to accomplish the same goal has been proposed by SALTZMAN and VERNEKAR [2], but their representation does not include the seasonal variation.

In the present paper the investigation has been limited to the case in which the meridional variation of the various dependent variables is prescribed as a single sinuoidal component. We are therefore not attempting to predict the detailed distribution of the zonally averaged quantities, but are mainly interested in the annual variation.

The model is capable of predicting maximum zonal winds of the correct order of magnitude and with a correct phase angle (*i.e.* time of maximum in the annual variation). The same statement can be made for the meridional transport of sensible heat. The momentum transport predicted by the model is also of the correct order of magnitude both with respect to the annual mean and the amplitude of the first Fourier component, but the predicated phase angle increases with decreasing pressure while the opposite variation is found from observational studies. The main discrepancy between the predicted annual variation of energy quantities in the model and in the atmosphere is that the predicted quantities show a too large annual variation, while the annual mean values and the phase angles are as correct as one can expect from this simplified approach to the problem.

The weakest part of this approach to the annual variation of the general circulation is undoubtedly the use of empirically derived exchange coefficients. The use of an exchange coefficient rests on the assumption that the transport is directed from regions of high to regions of low values of the zonally averaged parameters. It is known that this assumption becomes invalid for the heat transport in the lower stratosphere [3] and for the transport of geostrophic potential vorticity in the lowest part of the troposphere [7]. It seems therefore that this approach is limited to the major part of the tropospheric flow.

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