556.556.4

AN APPROXIMATE METHOD FOR THE CALCULATION OF THE PERIODS OF SEICHES

by

I. SALA

Helsinki University of Technology

Abstract

A method for calculating the periods of free oscillations in a closed basin of oblong form is described. The equation of motion (6), (3) is put in the form of an integral equation (8). For the approximate calculation of the eigen-value this equation is replaced by a linear homogeneous equation system (12). Finally the eigen-value can by the aid of a computer be obtained from the matrix equation (15). Approximate values for the main period and for the first few harmonics of lake Pyhäjärvi in Finland have been calculated. The distribution of the vertical displacement along the lake for the uninodal seiche is presented in Fig. 1. The position of the nodal-line and the values of the periods received here agree very well with those obtained by the conventional methods.

1. *Introduction*

When the water-masses in a lake basin, once the equilibriun is disturbed, oscillate back into equilibrium position, we observe periodic fluctuations of the water-level. These oscillations are known as »Seiches», DEFANT [1].

The problems connected with free oscillations (seiches) in closed basins of an oblong form but of variable width and depth have been the subject of very complete theoretical investigations. Almost all of these theoretical considerations only apply for such basins where there are no components of the horizontal motion in a direction perpendicular to the long axis (x-axis) of the lake or to the so called »Talweg», a line connecting the lowest points of the bottom of the lake, respectively.

2. *Fundamental equations of seiches*

The equation of motion has the form

$$\frac{\partial^2 \xi}{\partial t^2} = - g \frac{\partial \eta}{\partial x} , \tag{1}$$

where g is the acceleration of gravity and ξ is the horizontal displacement for the water particles in the x-direction, at the time t , when the ordinate of the free surface corresponding to the abscissa, at the time t , is noted by $h + \eta$, where h is the ordinate in the undisturbed state. The equation of continuity can be written in the form

$$\eta = - \frac{1}{b(x)} \frac{\partial}{\partial x} (S(x)\, \xi) \tag{2}$$

if $b(x)$ and $S(x)$ are respectively the variable width at the surface and the variable cross-section of the lake at the point x . The boundary conditions, neglecting the friction, are

$$\xi = 0 \quad \text{for } x = 0 \quad \text{and } x = l , \tag{3}$$

where l is the length of the lake.

If we introduce η from the equation of continuity into the equation of motion we obtain

$$\frac{\partial^2 \xi}{\partial t^2} = g \frac{\partial}{\partial x}\left(\frac{1}{b(x)} \frac{\partial}{\partial x} (S(x)\xi)\right) \tag{4}$$

which if

$$\xi = r(x) \sin (\omega t + \varepsilon) \tag{5}$$

is transformed into

$$\frac{\partial}{\partial x}\left(\frac{1}{b(x)} \frac{\partial}{\partial x} (S(x)\xi)\right) + \frac{\omega^2}{g} \xi = 0 \tag{6}$$

with the same boundary conditions as before.

3. *Solution of the eigen-value problem*

a. If we put

$$y(x) = S(x)\,\xi \tag{7}$$

our problem can be stated with the aid of an integral expression as follows:

$$y(x) = \frac{\omega^2}{g} \int_0^l K(x\,,t)\,f(t)\,dt\,, \tag{8}$$

where

$$f(t) = \frac{y(t)}{S(t)} \tag{9}$$

and the kernel

$$K(x\,,t) = \begin{cases} u(t)\left(1 - \dfrac{u(x)}{u(l)}\right), & \text{for } 0 \le t \le x \\[2ex] u(x)\left(1 - \dfrac{u(t)}{u(l)}\right), & \text{for } x \le t \le l \end{cases} \tag{10}$$

with

$$u(x) = \int_0^l b(t)\,dt\,. \tag{11}$$

For the approximate calculation of the eigen-value ω the graph of the function $K(x\,,t)\,f(t)$ is assumed to be replaced by an inside broken line the number of sides of which is 2^n where n is an integer. The argument values t_i which correspond to the corners of the broken line are located between 0 and l equidistant, STIEFEL [4]. The area of the surface under the graph, which is the same as the value of the definite integral in question, is replaced by the sum of the trapezoids and the equation (8) is then replaced by a linear homogeneous equation system as follows, SALA [2] and [3]:

$$\begin{cases} \lambda y_i = A_0 K(x_i\,,0)\,f(0\,,y_0) + A_1 K(x_i\,,t_1)\,f(t_1\,,y_1) + \\ \qquad A_2 K(x_i\,,t_2)\,f(t_2\,,y_2) + \ldots + A_{2^n} K(x_i\,,l)\,f(l\,,y_l) \\ i = 0, 1, 2, \ldots, 2^n \end{cases} \tag{12}$$

where

$$\lambda = \frac{g}{\omega^2 l} \tag{13}$$

and

$$x = \frac{il}{2^n} ;\, A_0 = A_{2^n} = \frac{1}{2^{n+1}} ;\, A_1 = A_2 = \ldots = A_{2^n-1} = \frac{1}{2^n} \tag{14}$$

By equating the determinant of that equation system to null we get an equation whose roots are the approximate values of the natural frequencies ω of the lake in question. One can, moreover, obtain better approximate values for ω from the equation system (12) with two or more successive values for n. Thus one can also get an estimate of the accuracy attained.

The problem in question can be put in the form of a matrix equation

$$\lambda \mathbf{y} = \mathbf{BDy} . \tag{15}$$

There $\mathbf{B}$ is the matrix whose elements are the AK-values in the equation system (12). This matrix is symmetric while $\mathbf{D}$ is a diagonal matrix of the values $1/S_i$.

b. If we like Chrystal [1] introduce in the equation (4) the variables (5), (7) and (11) and if we put

$$S(x)\, b(x) = v(x) \tag{16}$$

we obtain

$$\frac{d^2y}{du^2} + \frac{\omega^2}{gv}\, y = 0 \tag{17}$$

with the boundary conditions

$$y = 0 \text{ for } u = 0 \text{ and } u = a , \tag{18}$$

where a is the entire surface of the lake.

The eigen-value problem (17), (18) can be put into the integral form (8) where now, if the integration interval is normalized to the value $a = 1$, the kernel is

$$K(u\,,\,t) = \begin{cases} t(1-u)\,, & \text{for } 0 \leq t \leq u \\ (1-t)u\,, & \text{for } u \leq t \leq 1 \end{cases} \tag{19}$$

and

$$f(t) = \frac{y(t)}{v(t)} . \tag{20}$$

4. *An application of the method presented above*

Phil.lic. Pentti Mälkki has in his treatise[1]) on seiches of lake Pyhäjärvi in Finland obtained an average value of $T_1 = 95.1$ min for the main period and $T_2 = 50.7$ min and $T_3 = 36.4$ min for the first two harmonics, respectively. These values have been calculated with the conventional methods. Mr. Mälkki has kindly given me the material of his investigation (Table 1) and I have treated them with the method presented above.

Table 1. Values of the length x, the width b, the cross-sectional area S and the surface area u of lake Pyhäjärvi in Finland.

$\frac{x}{\text{km}}$	$\frac{b}{\text{km}}$	$\frac{S}{\text{m}^2}$	$\frac{u}{\text{km}^2}$
0	2	0	0
1	4.360	18400	3.840
2	4.840	22800	8.460
3	5.780	31600	14.100
4	6.160	33600	20.250
5	5.860	32000	26.190
6	5.680	30800	31.930
7	6.160	33200	37.860
8	7.260	37600	45.080
9	7.720	39200	52.710
10	7.920	36400	60.550
11	7.760	39200	68.350
12	8.320	42400	76.480
13	9.000	47600	84.970
14	9.200	49200	94.150
15	7.700	48800	102.410
16	7.140	47200	109.920
17	7.640	58800	117.640
18	7.260	51200	124.980
19	6.540	41600	131.650
20	6.000	29200	137.900
21	5.240	21600	143.240
22	3.500	14000	147.400
23	2.200	6400	150.010
24	1.520	2400	151.760
25	0.460	800	152.314
25.2	> 0	0	152.360

[1]) lic.-exam. at the University of Helsinki.

a. When we at first use the method in point 3 **a** and choose

$$\lambda = \frac{2^n g \bar{S}}{\omega^2 l a}, \tag{21}$$

where $\bar{S}$ is a constant, e.g. the area of the biggest cross-section of the lake basin, and a is the entire surface of the lake, we get the lower triangle of the symmetric matrix **B** in the form

(22)

The lower triangle of the symmetric matrix **B**

$$\begin{bmatrix} a_1(1-a_1) & & & & \\ a_1(1-a_2) & a_2(1-a_2) & & & \\ a_1(1-a_3) & a_2(1-a_3) & & & \\ \vdots & \vdots & & & \\ a_1(1-a_i) & a_2(1-a_i) \dots & a_i(1-a_i) & & \\ \vdots & \vdots & \vdots & & \\ a_1(1-a_{2^n-1}) & a_2(1-a_{2^n-1}) & a_i(1-a_{2^n-1}) & \dots & a_{2^n-1}(1-a_{2^n-1}) \end{bmatrix}$$

with the dimensionless parameter

$$a_i = \frac{u_i}{a} \tag{23}$$

and the diagonal matrix

$$\mathbf{D} = \begin{bmatrix} s_1 & & & & & & 0 \\ & s_2 & & & & & \\ & & s_3 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ 0 & & & & & & s_{2^n-1} \end{bmatrix} \tag{24}$$

where

$$s_i = \frac{\bar{S}}{S_i} \tag{25}$$

also is a demensionless quantity.

In the case of lake Pyhäjärvi the length of the lake basin is $l = 25.2$ km, the entire surface of the lake is $a = 152.36$ km^2, the constant $\bar{S} =$

58800 m² and the acceleration of gravity $g = 9.82$ m/s². According to the equation (21) then

$$\lambda = \frac{2^n \cdot 9.82 \cdot 5.88}{\omega^2 s^2 \cdot 2.52 \cdot 1.5236 \cdot 10^8}$$

and the period of the oscillation in minutes is

$$T = \frac{2\pi}{60\omega} = \frac{k}{\sqrt{2^n}} \sqrt{\lambda}\,; \; k = 270.0356\,. \tag{26}$$

If we choose $n = 1$ equation (15) gives

$$\lambda y_1 = \frac{u_1}{a}\left(1 - \frac{u_1}{a}\right) \frac{\bar{S}}{S_1}\, y_1$$

where $u_1 = 81.574$ km² is the area of the surface of the lake from section 0 to the middle section $(x = l\,/\,2)$ and $S_1 = 45520$ m² is the area of the cross-section of the lake basin at the point $x = l\,/\,2$. Thus we get $\lambda = 0.5354030 \cdot 0.4645970 \cdot 1.291740 = 0.3213159$ and $T = 108.236$ min $\simeq$ 108.2 min. The value $n = 2$ gives for the main period the approximate value $T_1 = 99.7$ min and for the first two harmonics the values $T_2 = 53.8$ min and $T_3 = 43.8$ min, respectively.

With the aid of our method the values of $T_1 = 96.48$ min, $T_2 = 47.89$ min and $T_3 = 35.93$ min are obtained when the length of the lake was divided into 8 equal parts *i.e.* $n = 3$, and with $n = 4$ the values 95.54 min, 46.56 min and 33.98 min, respectively. There is good reason to suppose that the sequence of the approximate values given with the successive values of n for the same period converges linearly with the ratio 1/4. Thus the expression $T_n + \frac{1}{3}(T_n - T_{n-1})$ gives the improved approximate value $T = 95.2$ min for the main period.

The values for u and S used by the author were, with a modulus of 1 km in the x-direction, determined by Mälkki with the aid of maps. Therefore numerous interpolations had to be used in this investigation. When the values of u and S are from the beginning determined at the points $x_i = \frac{il}{2^n}$ one only needs a minimum amount of preliminary calculations to get the matrices **B** and **D** in the equation (15).

b. The method in point 3 **b** with

$$\lambda = \frac{2^{3n}\, g\bar{v}}{\omega^2 a^2}$$

gives for the lower triangle of the symmetric matrix **B** the following form

The lower triangle of the symmetric matrix **B**

$$\begin{bmatrix} 2^n-1 \\ 2^n-2 & 2(2^n-2) \\ 2^n-3 & 2(2^n-3) & 3(2^n-3) \\ \vdots & \vdots & \vdots & \ddots \\ 2^n-i & 2(2^n-i) & 3(2^n-i) & & i(2^n-i) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 3 & 2\cdot 3 & 3\cdot 3 & \dots & i\cdot 3 & \dots \\ 2 & 2\cdot 2 & 3\cdot 2 & \dots & i\cdot 2 & \dots & (2^n-2)\cdot 2 \\ 1 & 2\cdot 1 & 3\cdot 1 & \dots & i\cdot 1 & \dots & (2^n-2)\cdot 1 & 2^n-1 \end{bmatrix} \tag{28}$$

and the diagonal matrix

$$\mathbf{D} = \begin{bmatrix} \frac{\bar v}{v_1} & & & & & & 0 \\ & \frac{\bar v}{v_2} \\ & & \frac{\bar v}{v_3} \\ & & & \ddots \\ & & & & \frac{\bar v}{v_i} \\ & & & & & \ddots \\ 0 & & & & & & \frac{\bar v}{v_{2^n-1}} \end{bmatrix} \tag{29}$$

If we choose $n = 2$

$$\mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \tag{30}$$

and with the value $n = 3$

$$\mathbf{B} = \begin{bmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 12 & 10 & 8 & 6 & 4 & 2 \\ 5 & 10 & 15 & 12 & 9 & 6 & 3 \\ 4 & 8 & 12 & 16 & 12 & 8 & 4 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 \\ 2 & 4 & 6 & 8 & 10 & 12 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix} \tag{31}$$

and when $n = 4$

$$\mathbf{B} = \begin{bmatrix} 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 14 & 28 & 26 & 24 & 22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 13 & 26 & 39 & 36 & 33 & 30 & 27 & 24 & 21 & 18 & 15 & 12 & 9 & 6 & 3 \\ 12 & 24 & 36 & 48 & 44 & 40 & 36 & 32 & 28 & 24 & 20 & 16 & 12 & 8 & 4 \\ 11 & 22 & 33 & 44 & 55 & 50 & 45 & 40 & 35 & 30 & 25 & 20 & 15 & 10 & 5 \\ 10 & 20 & 30 & 40 & 50 & 60 & 54 & 48 & 42 & 36 & 30 & 24 & 18 & 12 & 6 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 56 & 49 & 42 & 35 & 28 & 21 & 14 & 7 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 56 & 48 & 40 & 32 & 24 & 16 & 8 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 & 54 & 45 & 36 & 27 & 18 & 9 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 & 60 & 50 & 40 & 30 & 20 & 10 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 & 50 & 55 & 44 & 33 & 22 & 11 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 & 40 & 44 & 48 & 36 & 24 & 12 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 & 30 & 33 & 36 & 39 & 26 & 13 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 14 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{bmatrix} \tag{32}$$

In spite of the fact that matrix **B** is the same for any one of all the lake basins the solution of our proble based on the equation (17) is inferior to the solution based on the equation (6) in point 3 **a**. It is so because the calculation of the element in the diagonal matrix (29) is rather laborious and because of the fact that there is u instead of x as the argument so it may cause some inaccuracies in the results. Calculations with this method have not been made by the author.

5. *The vertical displacements of the seiches*

The relative magnitude of the vertical displacement, caused by the oscillation, can be calculated in the following way. Equations (2) and (7) give

$$\eta = -\frac{1}{b(x)}\frac{d(y(x))}{dx}\,. \tag{33}$$

The Electronic Digital Computer gives in the form of the sequence $y_1, y_2, y_3, \ldots, y_i, \ldots, y_{2^n-1}$ the relative values of the eigen-vector. If we put into the equation (33) the values b_i and substitute for $\frac{dy}{dx}$

the expressions $\frac{\Delta y}{\Delta x} = \frac{1}{2\Delta x}(y_{i+1} - y_{i-1})$ and take into consideration that $\Delta x = x_i - x_{i-1}$ is constant and therefore has no effect upon the relative values of η we get

$$\eta_i = -\frac{y_{i+1} - y_{i-1}}{2b_i}\,; \; i = 1, 2, 3, \ldots, 2^n - 1\,.$$

The curve in Fig. 1 presents the relative vertical displacement η of lake Pyhäjärvi according to the main period and obtained by the method mentioned above. As it can be revealed the nodal-line is located at the point $x \simeq 0.4\,l$. The value $n = 4$ gives $x \simeq 0.42\,l$. They are in good accordance with the value $x \simeq 0.44\,l$ obtained by Mälkki.

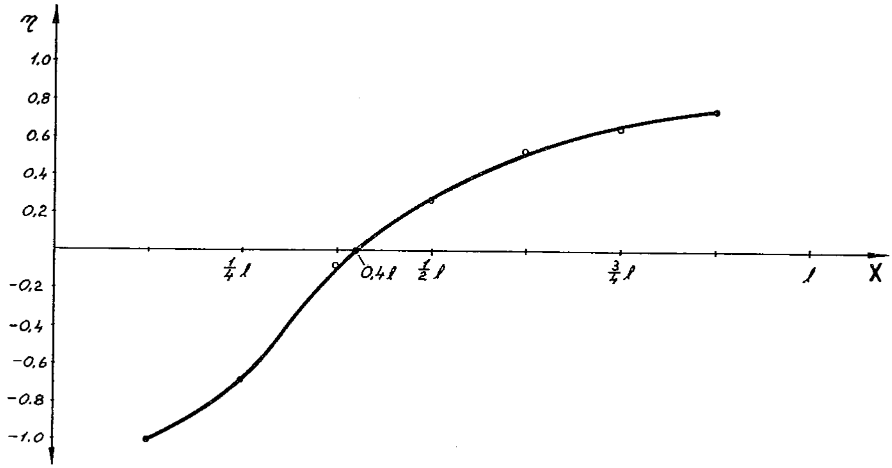

Fig. 1. Relative vertical displacement η according to the uni-nodal seiche in lake Pyhäjärvi in Finland.

REFERENCES

1. Defant, Albert, 1960: *Physical Oceanography,* Vol. 2, Pergamon Press, pp 160–173
2. Sala, I., 1950: Numerische Lösung von Linearen homogenen Eigenwertaufgaben zweiter Ordnung durch Mittelwertmethoden. *Soc. Scient. Fenn., Comm. Phys-Math.* XV. 13, Helsinki.
3. —»— 1963: On the numerical solution of certain boundary value problems and eigenvalue problems of the second and the fourth order with the aid of integral equations. *Acta Polyt. Scand. Math.* and *Comp. Machin. Ser.* No. 9, Helsinki.
4. Stiefel, E., 1961: Altes und Neues über numerische Quadratur. *ZAMM* **41,** 408–413.