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ON THE DIFFUSIVITY OF TURBULENT FLOW

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A particle in a turbulent flow is moving at random. Its new position after any interval can not be predicted. In such a case the Calculus of Probability is used, and it is assumed that the new position of the particle is a stochastic variable. In this particular case the above is not strictly true, and then the transition is slightly dependent upon the stream configuration. We consider the material (salt, dye etc.) to be normally distributed in the neighborhood of the point (s_1^0, s_2^0, s_3^0) with the dispersions $\sigma_1, \sigma_2, \sigma_3$ corresponding to the s_1, s_2, s_3 axes so that we have the initial concentrations $(t = 0)$

$$C(s', 0) = A \prod_{\nu=1}^{3} \frac{e^{-\frac{(s_\nu' - s_\nu^0)^2}{2\sigma_\nu^2}}}{\sqrt{2\pi}\,\sigma_\nu}, \tag{1}$$

where A is a constant.

We assume that the mean velocity of the stationary current is $\boldsymbol{v} = (v_1, v_2, v_3)$. Let the initial position of the particle be $\boldsymbol{s}' = (s_1', s_2', s_3')$ and after a short interval Δt be $\boldsymbol{s} = (s_1, s_2, s_3)$. We write

$$\boldsymbol{r} = (r_1, r_2, r_3) = \boldsymbol{s}' - \boldsymbol{v}\, t - \boldsymbol{s}\,. \tag{2}$$

The transition of the material from the volume element dV' at $\boldsymbol{s}'$ to $\boldsymbol{s}$ during the time interval Δt may have the probability $\varphi dV'$. We assume that φ is a normal distribution function

$$\varphi = \varphi(\boldsymbol{r}, \Delta t) = \prod_{\nu=1}^{3} \frac{e^{-\frac{r_\nu^2}{2\tau_\nu^2}}}{\sqrt{2\pi}\,\tau_\nu}, \tag{3}$$

where the dispersions τ_ν depend on time interval Δt. Then we get the concentration at the point $\boldsymbol{s}$ at the time Δt

$$C(\boldsymbol{s}, \Delta t) = \int C(\boldsymbol{s}', 0)\varphi(\boldsymbol{r}, \Delta t) dV' , \tag{4}$$

where the integral is to be calculated over the whole volume considered.

When v_ν and σ_ν are constants it can be shown using (1) and (3) that

$$C(\boldsymbol{s}, \Delta t) = A \prod_{\nu=1}^{3} \frac{e^{-\frac{(s_\nu - s_\nu{}^0 - v_\nu \Delta t)^2}{\sigma_\nu{}^2 + \tau_\nu{}^2}}}{\sqrt{2\pi(\sigma_\nu^2 + \tau_\nu^2)}} . \tag{5}$$

To gain the time dependence of τ_ν, we start with $C(\boldsymbol{s}, \Delta t)$ and repeat the procedure n times and get

$$C(\boldsymbol{s}, \Delta t) = A \prod_{\nu=1}^{3} \frac{e^{-\frac{(s_\nu - s_\nu{}^0 - n v_\nu \Delta t)^2}{\sigma_\nu{}^2 + n\tau_\nu{}^2}}}{\sqrt{2\pi(\sigma_\nu^2 + n\tau_\nu^2)}} . \tag{6}$$

By writing $n\Delta t = t$ and denoting $\frac{\tau_\nu^2}{\Delta t} = \varkappa_\nu$ we determine $\varphi(\boldsymbol{r}, t)$ and $C(\boldsymbol{s}, t)$ from equations (3) and (6).

$$\varphi(\boldsymbol{r}, t) = \prod_{\nu=1}^{3} \frac{e^{-\frac{r_\nu{}^2}{2\varkappa_\nu t}}}{\sqrt{2\pi \varkappa_\nu t}}, \tag{7}$$

$$C(\boldsymbol{s}, t) = A \prod_{\nu=1}^{3} \frac{e^{-\frac{(s_\nu - s_\nu{}^0 - v_\nu t)^2}{\sigma_\nu{}^2 + \varkappa_\nu t}}}{\sqrt{2\pi(\sigma_\nu{}^2 + \varkappa_\nu t)}} . \tag{8}$$

Thus $\varkappa_1, \varkappa_2, \varkappa_3$ are constants.

When we calculate the concentration in the case, where v_ν and $\varkappa_\nu$ vary in space, the equation (4) with the distribution function (7) is to be used, but the evaluation of the integral even approximately is cumbersome. In principle it is possible to calculate the concentration in the nonstationary case also, by using (4) repeatedly with short intervals Δt and taking the limiting value as Δt approaches zero.

We wish to consider a simple case in which the material initially is concentrated at a single point, and we assume further that the flow is stationary and uniform. Then the point of strongest concentration according to (8) is $\boldsymbol{s} = \boldsymbol{s}^\circ + \boldsymbol{v}t$, and we have

$$C(\boldsymbol{s}, t) = A \prod_{\nu=1}^{3} (2\pi\varkappa_\nu)^{-\frac{1}{2}} t^{-\frac{3}{2}} . \tag{8$'$}$$

The maximum concentration is thus proportional to the power $t^{-\frac{3}{2}}$.

When we instead of a single point take a homogeneous line of material of given length, then at first the maximum concentration is proportional to the power t^{-1}, for in the direction of the line the effect of the diffusion will not be noticeable at all and the effect of the corresponding factor falls out. Later on, however, the concentration behaves more like (8)$'$.

In the nonstationary case again the diffusion is stronger than the formulas (8) and (8)$'$ are predicting.