Geophysica (1985), Vol. 21, No. 1, pp. 39–50.
Printed in Finland

550.343

SEISMIC HAZARD IN FINLAND: EVALUATION OF M_{max}

by

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Abstract

The truncated Gutenberg-Richter frequency-magnitude relationship in conjunction with a Poisson distribution is used to develop an asymptotic distribution of extreme values. This model is applied to estimate M_{max} and to estimate the probability of occurrence of earthquakes in Finland, using for earthquakes occurring throughout the period 1700–1979. The earthquake with the greatest magnitude within a ten-year period was used. The application of maximum likelihood procedure for the parameter estimation provides $M_{max} = 5.0 \pm 0.1$.

1. *Introduction*

EPSTEIN and LOMNITZ (1966) have shown that Gumbel's statistical model 1 of extreme magnitudes (GUMBEL, 1958) can be derived directly from the assumptions that earthquakes are generated by a simple Poisson process and they follow the well-known frequency – magnitude relation:

$$\log N = a - bm \tag{1}$$

Their paper has recalled the first application of Gumbel's statistics in seismology (NORDQUIST, 1945) and has started an »avalanche» of applications of extreme value methods for earthquake hazard estimation. The extensive lists of references on this subject can be found in recent papers (*e.g.* KNOPOFF and KAGAN, 1977; WEICHERT and MILNE, 1979). Moreover, by deriving the first Gumbel's distribution from commonly accepted rules related to earthquake occurrence, EPSTEIN and LOMNITZ (1966) have established a »kind of physical basis» for applications of this distribution in earthquake statistics. However, soon after publication of their paper, a number of authors noticed that the application of the Gutenberg-Richter frequency-magnitude formula with unbounded argument results in an unbounded distribution of extremes. Since, from a physical point of view, there must exist at least an upper limit to the earthquake magnitude (*e.g.* YEGULALP and KUO, 1974; KNOPOFF and KAGAN, 1977), the first Gumbel distribution of extreme magnitudes has only an approximate character. To avoid this contradition some authors proposed the so-called third Gumbel distribution for which an upper limit for the magnitude is introduced. Regardless of the fact that the third Gumbel distribution has been used in seismological rather successfully, it cannot be derived from the rules of earthquake occurrence.

In this paper, an alternative distribution of the maximum magnitude value is proposed, in which an upper limit for the magnitude is assumed. This new distribution is not just another empirical relation (*e.g.* HOWELL, 1981), but is derived from the commonly accepted assumptions related to earthquake occurrence.

2. *Theoretical background*

Let us assume that

(a) the annual number of earthquakes is a Poisson random variable with the mean λ

(b) the earthquake magnitude M is a random variable, distributed according to a double truncated exponential probability cumulative distribution

$$F(m) \equiv Pr(M \leqslant m) = \frac{A_1 - A(m)}{A_1 - A_2}, \tag{2}$$

where $A_1 = \exp(-\beta M_{\min})$, $A_2 = \exp(-\beta M_{\max})$, $A(m) = \exp(-\beta m)$, $M_{\min}$ is the threshold magnitude value, $M_{\max}$ is the maximum regional magnitude value, and β is the parameter.

Assumption (a) is certainly acceptable if aftershock magnitudes are eliminated from the statistics (*e.g.* LOMNITZ, 1966; RADU, 1973; POWELL and DUDA, 1975).

It is also interesting to note that the Poissonian temporal distribution of events results from the principle of maximum entropy (TRIBUS, 1969, BERRILL and DAVIS, 1980).

As to assumption (b), the truncated exponential frequency – magnitude relation (2) is well known in seismology. It was proposed by CORNELL and VANMARCKE (1969), and later was obtained by COSENTINO *et al.* (1977) who introduced a simple model based on a number of assumptions, among which was the existence of the maximum regional magnitude value M_{max}. The same distribution was obtained by BERRILL and DAVIS (1980) from the principle of maximum entropy. Relation (2) agrees well with observed data, both at small magnitudes where it coincides with the Gutenberg-Richter relation, and at large magnitudes where it fits the data remarkably well.

From assumptions (a) and (b) it follows that the largest selected in certain of time earthquake magnitude is distributed according to the following cumulative distribution:

$$G(x) \equiv Pr(X \leqslant x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(\frac{A(x) - A_1}{A_2 - A_1}\right)^k , \tag{3}$$

denoting $\lambda \left(\frac{A(x) - A_1}{A_2 - A_1}\right) = \Lambda$, and since

$$\sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} = \exp(\Lambda).$$

we obtain the new distribution of extremes as follows (KIJKO, 1982; 1983).

$$G(x) = \exp\left[-\lambda\left(\frac{A_2 - A(x)}{A_2 - A_1}\right)\right]. \tag{4}$$

The resulting cumulative probability distribution (4) is doubly truncated. The first truncation M_{min} in the formula $A_1 = \exp(-\beta M_{min})$ represents the chosen threshold magnitude. The second truncation M_{max} in the formula $A_2 = \exp(-\beta M_{max}$ is an unknown parameter representing the maximum possible magnitude in a given region. From the definition of A_1 and A_2 it follows that for $M_{max} \to \infty$, $A_2 \to 0$ and for $M_{min} = 0$, $A_1 = 1$. Thus for $A_1 = 1$ and $A_2 = 0$, the new derived distribution (4) becomes:

$$G(x) = \exp[-\lambda \exp(-\beta x)]. \tag{5}$$

Since the probability distribution (5) is well known as the first Gumbel distribution, the new distribution (4) can be called truncated first Gumbel distribution of the largest values.

3. *Parameters estimation of distribution*

In order to estimate the parameters $\bar{\theta} = (\beta, \lambda, M_{\max})$, the largest earthquake magnitudes $\bar{X} = (X_1, \ldots, X_N)$ are selected from N consecutive time intervals, and the maximum likelihood method is used. The likelihood function $L(\bar{\theta}/\bar{X})$ is given by:

$$L(\bar{\theta}/\bar{X}) = \prod_{i=1}^{N} g(X_i), \tag{6}$$

where $g(x)$ is the density distribution of the form:

$$g(x) = \exp\left[\ln G(x) + \ln \frac{\beta\lambda}{A_1 - A_2} - \beta x\right]. \tag{7}$$

The partial derivatives of $\ln L$ with respect to β and λ are:

$$\frac{\partial \ln L}{\partial \beta} = \frac{M_{\max} A_2 - \langle XA \rangle}{A_2 - A_1} + \frac{1}{\beta} - \langle X \rangle, \tag{8}$$

$$\frac{\partial \ln L}{\partial \lambda} = -\frac{A_2 - \langle A \rangle}{A_2 - A_1} + \frac{1}{\lambda},$$

where

$$\langle X \rangle = \sum_{i=1}^{N} \frac{X_i}{N},$$

$$\langle A \rangle = \sum_{i=1}^{N} \frac{A(X_i)}{N} = \sum_{i=1}^{N} \frac{\exp(-\beta X_i)}{N},$$

$$\langle XA \rangle = \sum_{i=1}^{N} X_i \frac{A(X_i)}{N}.$$

Placing $\frac{\partial \ln L}{\partial \lambda} = 0$, we obtain

$$\lambda = \frac{A_2 - \langle A \rangle}{A_2 - A_1}. \tag{9a}$$

Placing $\frac{\partial \ln L}{\partial \beta} = 0$ and substituting λ by relation (9a), we have

$$\frac{1}{\beta}=\frac{\langle XA\rangle - M_{max}A_2}{A_2-\langle A\rangle}+\langle X\rangle . \tag{9b}$$

Formulas (9a) and (9b) provide two equations which can be used for the maximum likelihood estimation of β and λ. It should be noted that for the $M_{max}\to\infty$ relation, (9b) is reduced to the maximum likelihood estimation of the parameter β of Gumbel 1 distribution (KIMBALL, 1946). It should be also noted that equation (9b) is independent of the parameter λ.

The likelihood function (6) decreases monotonically for $M_{max}\to\infty$, leading to no maximum likelihood estimation for M_{max}. Therefore, the M_{max} estimation can only be carried out by the introduction of some additional equation. It seems reasonable to assume that the information about M_{max} is provided by maximum values of the observed magnitude X_{max}, where $X_{max} = \max(X_i)$, $i = 1,...,N$.

Thus, in order to estimate M_{max}, the new constraint should be built established in such a way that the value of X_{max} plays the leading role. Such a condition can be established as follows.

By definition, the cumulative distribution function for the largest earthquake magnitude in N time intervals is of the form

$$G^{\star}(x_{max}) = G^N(x). \tag{10}$$

This, in the discussed case

$$G^{\star}(x_{max}) = \begin{cases} \exp\left[-\lambda N\left(\frac{A_2-A(x_{max})}{A_2-A_1}\right)\right], & M_{min}\leqslant x_{max}\leqslant M_{max} \\ 1 & , M_{max}< x_{max} \end{cases} \tag{11}$$

and all the characteristics of x_{max} can be obtained from similar formulas, simply multiplying λ by N. After application of the moment generating function, it was shown by KIJKO (1983) that the expected largest magnitude in one time interval is

$$E(x) = M_{max} - \frac{E_1(z_2)-E_1(z_1)}{\beta\exp(-z_2)}, \tag{12}$$

where $z_1 = \lambda_1\cdot A_1$, $z_2 = \lambda_1\cdot A_2$, $\lambda_1 = -\lambda/(A_2-A_1)$ and $E_1(\cdot)$ denotes an exponential integral function (ABRAMOWITZ and STEGUN, 1964)

$$E_1(z) = \int_z^{\infty}\exp(-\zeta)/\zeta\, d\zeta .$$

From relation (12), therefore, the expected largest magnitude in N time intervals is

$$E(X_{\max}) = M_{\max} - \frac{E_1(Nz_2) - E_1(Nz_1)}{\beta \exp(-Nz_2)}\ . \tag{13}$$

Introducing the following condition

$$X_{\max} = E(X_{\max}) \tag{14}$$

into relation (9a) and (9b) we obtain a set of equations determining the maximum likelihood solution. Since for real seismic data, $z_1 \cdot N > 1$ and $z_2 \cdot N > 1$, $E_1(z)$ can be expressed as (AMBRAMOWITZ and STEGUN, 1964)

$$E_1(z) = \frac{1}{z}\exp(-z)\frac{z^2 + a_1 z + a_2}{z^2 + b_1 z + b_2}\,, \tag{15}$$

where $a_1 = 2.334733$, $a_2 = 0.250621$, $b_1 = 3.330657$ and $b_2 = 1.681534$. Formula (15) is an approximation of the exponential integral function with a maximum error of $5 \cdot 10^{-5}$ for $1 \leqslant z < \infty$.

The set of equation (9) and (14) can be readily solved by an iterative procedure, even with the aid of a micro-computer.

4. *Standard errors of distribution parameters*

A formal estimate of the variance of $\hat{\theta} = (\hat{\beta}, \hat{\lambda}, \hat{M}_{\max})$ can be obtained from the relations describing the variance-covariance matrix of vector $\hat{\theta}$, estimated by the maximum likelihood method with constrained parameters (EADIE *et al.*, 1971):

$$\bar{D}(\hat{\theta}) = \bar{A}^{-1} - \bar{A}^{-1}\bar{B}(\bar{B}^T\bar{A}^{-1}\bar{B})^{-1}\bar{B}^T\bar{A}^{-1} \tag{16}$$

where in our case the matrices A and B are of the form

$$\bar{A} = \{a_{ij}\} = \frac{\partial^2 \ln L}{\partial\theta_i \partial\theta_j}\,, \quad i, j = 1, 2, 3. \tag{17}$$

$$\bar{B} = \{b_{ij}\} = \frac{\partial E(X_{\max})}{\partial\theta_i}\,, \quad i = 1, 2, 3, \; j = 1.$$

It should be noted that relations (16) and (17) do not indicate how significant the obtained estimation of the vector $\hat{\theta}$ is. It is clear that the maximum likelihood procedure derived here can give biased $\hat{\theta}$ parameters (especially $M_{\max}$), if it is applied to the biased X_{max}. It is well known that the uncertainly of the maximum observed magnitude X_{max} can be as high as 0.5 of the magnitude unit when it is based historical intensity data.

Let us discuss in detail the influence of a »wrong» value of X_{max} on the estimated M_{max}.

Taking into account that for real seismic data $E_1(z_1N) \ll E_1(z_2N)$, from relations (13) and (14) we have

$$\frac{dX_{max}}{dM_{max}} = 1 - \frac{1}{\beta}\left[e^{\xi}E_1(\xi) - \frac{1}{\xi}\right]\frac{d\xi}{dM_{max}}\,, \tag{18}$$

where

$$\frac{d\xi}{dM_{max}} = -\beta\xi\,\frac{A_1 - 2A_2}{A_1 - A_2} \tag{19}$$

and $\xi = Nz_2$. Ignoring the influence of uncertainities in the $\hat{\beta}$ and $\hat{\lambda}$ estimations after simple algebraic transformations, the approximate standard deviation of M_{max} becomes

$$\sigma M_{max} = T_c(\hat{\theta})\,\sigma_x \tag{20}$$

where

$$T_c(\hat{\theta}) = \mathrm{Abs}\left\{1 + [\xi e^{\xi}E_1(\xi) - 1]\,\frac{A_1 - 2A_2}{A_1 - A_2}\right\}^{-1} \tag{21}$$

and σ_x is the standard deviation of X_{max} · $T_c(\hat{\theta})$ can be considered as a »transmition coefficient» which transmits the uncertainities in X_{max} into the uncertainities of the M_{max} estimation.

When $A_2 \ll A_1$, which in practice means that the earthquake cataloque used covers a sufficiently long period and includes a wide range of magnitudes, the coefficient $T_c(\hat{\theta})$ takes the following form:

$$T_c(\hat{\theta}) = \mathrm{Abs}[\xi e^{\xi}E_1(\xi)]^{-1}. \tag{22}$$

An approximate value of the same transmition coefficient can also be obtained numerically. If after solving equations (9) and (14), $M_{max(1)}$, $M_{max(2)}$ are the estimations of M_{max} respectively for $X_{max(1)} = X_{max} - \delta$ and $X_{max(2)} = X_{max} + \delta$ where δ = a small disturbance (*e.g.* 0.1), then $T_c(\hat{\theta})$ is given approximately by

$$T_c(\hat{\theta}) = \mathrm{Abs}(M_{max(2)} - M_{max(1)})/2\delta\,. \tag{23}$$

5. *Earthquakes in Finland*

Finland is situated on the Precambrian Baltic shield in Fennoscandia. This area is seismically very peaceful. About 50–70 earthquakes with a magnitude of less than 6 are registered yearly. These earthquakes represent intraplate seismicity and have been attributed to Fennoscandian land uplift and to plate-tectonic forces from the North Atlantic Ridge towards Fennoscandia.

The Finnish earthquake cataloque (PENTTILÄ, 1982) reports 341 earthquakes throughout the period 1610–1980. Only one fifth of the events occurred prior to the 1850s, when systematic collecting of earthquake observations began in Finland, Prior to the 1950s, when short-period recording started in Fennoscandia, earthquake data was based on macroseismic observations. From 1951–1980, the period when sophisticated instrumentation was used, 113 Finnish earthquakes were registered. The magnitude estimates for Finnish earthquakes were computed using macroseismic observations, *i.e.* maximum intensity and radius of macroseismic area. BÅTH's method (1953) with a coefficient of attenuation = 4 and WAHLSTRÖM and AHJOS' (1982) macroseismic formula scaled to M_L were applied and the larger of the two magnitudes was the preferred reading for a single earthquake. Since 1960 the regional M_L-magnitude was computed using WAHLSTRÖM and AHJOS' method (1982). In Finland about 5–10 earthquakes are registered yearly. The magnitudes of Finnish events have varied between 1.5 and 4.9, nearly 80 % of events had a magnitude of less than 3 and only ten earthquakes had a magnitude equal to or greater than 4.5.

Table 1. Frequency distribution of maximum magnitudes in 10-year intervals for Finland for the period 1700–1979.

Magnitude	Frequence
2.5	1
2.7	2
2.8	1
3.2	3
3.4	2
3.6	2
3.7	1
3.8	2
3.9	2
4.0	3
4.2	1
4.5	3
4.6	1
4.7	2
4.9	1

Table 2. Estimates of the average return periods for earthquakes in Finland.

M	Average return periods (years)
3.0	11.5
3.25	13.0
3.5	15.4
3.75	19.3
4.0	25.7
4.25	37.1
4.5	60.4
4.6	77.9
4.7	107
4.8	162
4.9	310

The proposed procedure for an estimation of the distribution parameters was applied to earthquakes in Finland throughout the period 1700–1979. The earthquake with the greatest magnitude within a ten-year period was used (Table 1).

6. *Results*

The application of the maximum likelihood procedure with condition (14) to the data listed in Table 1 gives 1.14 ± 0.34, 3.73 ± 0.95 and 5.02 ± 0.07 as the estimates of $\hat{\beta} \pm \hat{\sigma}_\beta$, $\hat{\lambda} \pm \hat{\sigma}_\lambda$ and $\hat{M}_{max} \pm \hat{\sigma}_{M_{max}}$ (Fig. 1). Since the observed magnitudes are grouped, they are represented in Fig. 1 by the mean values of each group and the so-called Weibull plotting position was used (GUMBEL, 1958; MAKJANIC, 1980). Of course, the chosen plotting rule does not play any role in the estimation of the parameters.

The transmition coefficient calculated according to formula (20) is equal to 1.13. The same coefficient calculated numerically (eg. 23) is equal to 1.30. Thus for the standard deviation of the maximum observed magnitude $\sigma_x = 0.1$, with an accuracy to the first decimal digit, all three methods give the same value of $\hat{\sigma}_{M_{max}} = 0.1$. It is interesting to note that $M_{max} = 5.0 \pm 0.1$ found here is equal to the M_{max} estimated by the Gumbel third distribution (AHJOS *et al.*, 1984).

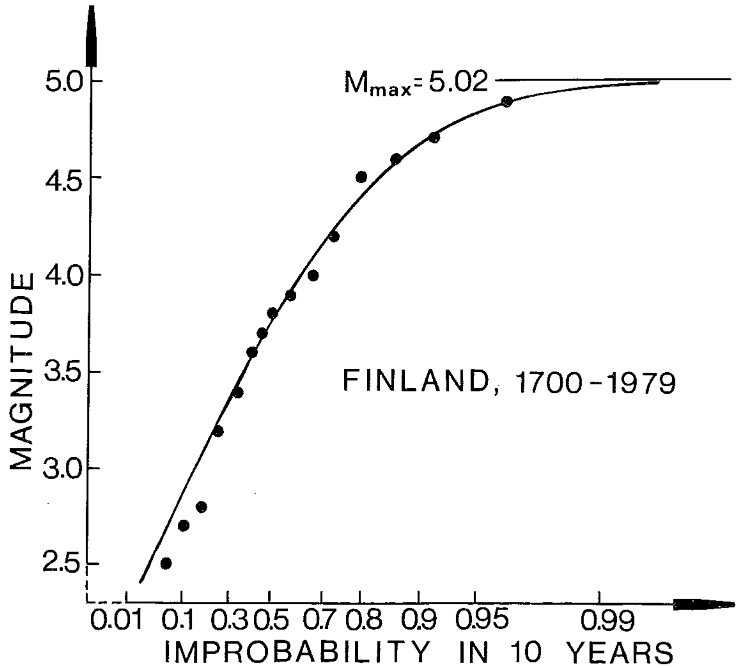

Fig. 1. Distribution of largest magnitudes in 10-year intervals for Finland between 1700 and 1979.

7. *Conclusions*

1. Introduction of a concept of maximum regional magnitude M_{max} to derive the distribution of the largest earthquake magnitudes provides a modified form of the first Gumbel distribution. For $M_{max} \to \infty$, the new distribution tends asymptotically to the first Gumbel distribution. It should be emphasized that the new proposed distribution is derived from the commonly accepted assumptions and constraints related to earthquake occurrence.

2. The described procedure permits the calculation of maximum likelihood estimates of the parameters of extreme magnitudes distribution. Considering the importance of M_{max} values for seismic hazard analyses, the additional formulas are given describing uncertainties of their estimates. Moreover, a slight modification of condition (14) provides an estimation of M_{max} based on the largest known historical earthquakes observed in a given area. This can be performed by substituting X_{max} by the largest known magnitude and N by the number of time intervals used (KIJKO, 1984).

3. The application of the described procedure to the largest magnitudes in ten-year intervals in Finland for the period 1700 – 1979 provides $M_{max} = 5.0 \pm 0.1$.

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