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ITERATIVE SOLUTION OF THE ELECTRODYNAMICAL INVERSION PROBLEM FOR VERTICALLY VARYING EARTH

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Abstract

An iterative method for solving the one-dimensional electrodynamical inversion problem of determining the conductivity and permeability of vertically varying earth from the surface reflection coefficient is developed. The method is a generalization of the Jost-Kohn method for inverting the Born series for the phase shift in quantum scattering theory. The convergence of the method is illustrated with an analytically solvable example. The convergence problem associated with the sharp surface boundary is circumvented by treating the surface discontinuity exactly.

1. *Introduction*

The electrodynamical geophysical inversion problem of determining the conductivity and permeability parameters of subterranean strata from surface reflection data is of formidable complexity even in the simplest case of vertically varying strata with no horizontal variation. For this case the inversion problem may in principle be solved by the methods developed by JAULENT (1976) and JAULENT and JEAN (1972) if the initial electromagnetic field is a vertical plane wave. The method of Jaulent and Jean is a generalization of the inverse scattering methods of GEL'FAND and LEVITAN (1951) and AGRANOVICH and MARCHENKO (1964) to the case of linearly frequency dependent potentials and requires solution of a set of coupled integral and differential equations. Because of the complexity of the Jaulent-Jean equations and the lack of practical algorithms for their solution we shall here develop an alternative simpler but iterative method for solving the geophysical inversion problem. This method is a generalization of the method of JOST and KOHN (1952) for determining the scattering potential from the phase shift in quantum scattering theory by iterative inversion of the Born series for the phase shift. The method proposed should be far more convenient to use in practical geophysical sounding work than solution of the Jaulent-Jean equations as it only involves quadrature of measured quantities.

The one-dimensional geophysical inversion problem is considerably more complicated than the corresponding inverse scattering problem in quantum theory because of the important role of attenuation which appears as an absorptive linearly frequency dependent potential. While this complexity is reflected in the Jaulent integral equation formulation (RISKA, 1981) it does not complicate the form of the Born series for the reflection coefficient in an essential way. The only problem associated with employment of iterative inversion of the Born series may be slow convergence or lack of convergence as in the case of scattering from discontinuous strata or sharp boundaries (PROSSER, 1976). Such discontinuities may appear in the geophysical case for example at the surface of the earth which may represent a discontinuous step in the conductivity and permeability parameters. To achieve convergence of the iterative method one in such cases has to treat the surface discontinuity exactly and develop the Born series around the solution of the exactly solved discontinuity problem.

The presently developed method of solving the geophysical electrodynamical inversion problem should be simple to use in practice as it reduces the problem to quadrature. The ultimate value of the method will however depend on the rate of convergence in the topmost layers. The method solves a more general problem than the approximative integral equation method previously developed by WEIDELT (1972) in which the displacement current term in the wave equation is neglected. That term has of course to be taken into account if any information at all is sought on the permeability or permittivity distributions. While neglecting the displacement current term is a common approximation in low frequency geophysical sounding work it cannot be dropped in the solution of the inverse scattering problem which involves the high frequency behaviour of the reflection coefficient as well as the low frequency behaviour.

This paper falls into 5 sections. In section 2 we formulate the inverse scattering problem for electromagnetic sounding. In section 3 we develop the iterative method for solving the problem in general. This method is illustrated with an analytically solvable example in Appendix 1. In section 4 we develop the model potential method for the case in which the surface discontinuity is treated exactly. A set of explicit formulae for use with the model potential method is given in Appendix 2. Finally section 5 contains a concluding discussion.

2. *The one-dimensional inversion problem*

An electromagnetic plane wave with harmonic time dependence propagating vertically downwards (Fig. 1) satisfies the wave equation

$$\left\{\frac{d}{dz}\frac{1}{\mu'(z)}\frac{d}{dz}+\frac{\omega^2\epsilon'(z)}{c^2}+i\frac{\omega\sigma(z)}{c^2\epsilon_0}\right\}E(z)=0. \tag{2.1}$$

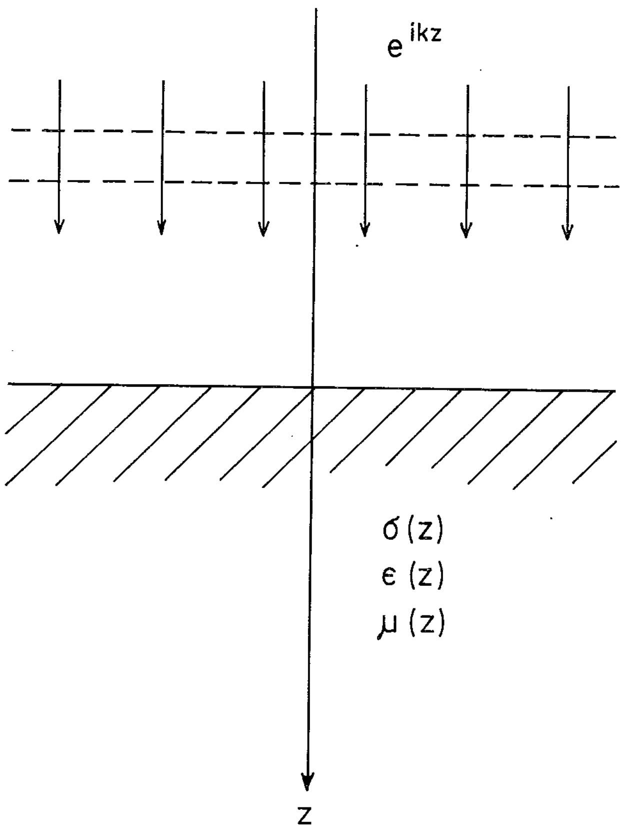

Fig. 1. Plane electromagnetic wave reflected from vertically layered earth.

Below ground ($z > 0$) μ' and ϵ' represent the relative magnetic and dielectric permeabilities and σ the conductivity. Above ground $\epsilon' = \mu' = 1$ and $\sigma = 0$. In (2.1) c is the velocity of light in vacuum and ω the angular frequency. The equation (2.1) may be reduced to standard Sturm-Liouville form by the Liouville transform (JAULENT (1976)):

$$x = \theta(z) \int_0^z dz' \sqrt{\epsilon'(z')\mu'(z')} + \theta(-z) z,$$
$$y(x) = \left(\frac{\epsilon'(x)}{\mu'(x)}\right)^{1/4} E(x). \tag{2.2}$$

Setting $k = \omega/c$ the transform (2.2) yields for y the Sturm-Liouville equation

$$\frac{d^2 y}{dx^2} + [k^2 - V(x, k)] y = 0, \tag{2.3}$$

with

$$V(x, k) = U(x) + ikQ(x). \tag{2.4}$$

The real and imaginary components of the potential are

$$U(x) = \left(\frac{\mu'(x)}{\epsilon'(x)}\right)^{1/4} \frac{d^2}{dx^2} \left(\frac{\epsilon'(x)}{\mu'(x)}\right)^{1/4}, \tag{2.5a}$$

$$Q(x) = -\frac{\sigma(x)}{c\epsilon_0 \epsilon'(x)}. \tag{2.5b}$$

The equation (2.3) differs from the common Sturm-Liouville equation in scattering theory by the complex eigenvalue dependent interaction (2.4).

The inversion problem consists of determining the potential $V(x, k)$ from the surface reflection coefficient for the electric field E and subsequent determination of the ratios σ/ϵ' and ϵ'/μ' from the potential V. If either ϵ' or μ' is known the potential V determines the conductivity profile and the unknown permeability quantity uniquely. On the other hand JAULENT (1976) has demonstrated that the surface reflection coefficient cannot determine both potential components U and V uniquely without additional assumptions. Sufficient additional assumptions would be to assume the presence of a perfectly reflecting layer at given depth d and to give the integrated value of the attenuating potential $Q(x)$. These parameters then characterize the solution. As we shall demonstrate below the parameter choices simply represent the method of extrapolating the reflection coefficient into the unmeasurable high frequency region. We shall therefore develop the inversion method with the assumption of the presence of a deep perfectly reflecting layer. The practical utility of the inversion solution will depend on to what extent the solution in the topmost region remains insensitive to the depth parameter d or equivalently the method of high frequency extrapolation.

3. *Iterative solution of the inversion problem*

We consider reflection of a plane wave electric (or magnetic) field y (2.2) from vertically varying earth with the assumption of the presence of a perfectly reflecting layer at depth $x = d$ as illustrated in Fig. 2. The surface is taken to be at $x = 0$. Following JAULENT (1976) we consider the pair of differential equations

$$\frac{d^2 y^{\pm}(x)}{dx^2} + [k^2 - V^{\pm}(x, k)]\, y^{\pm}(x) = 0, \tag{3.1}$$

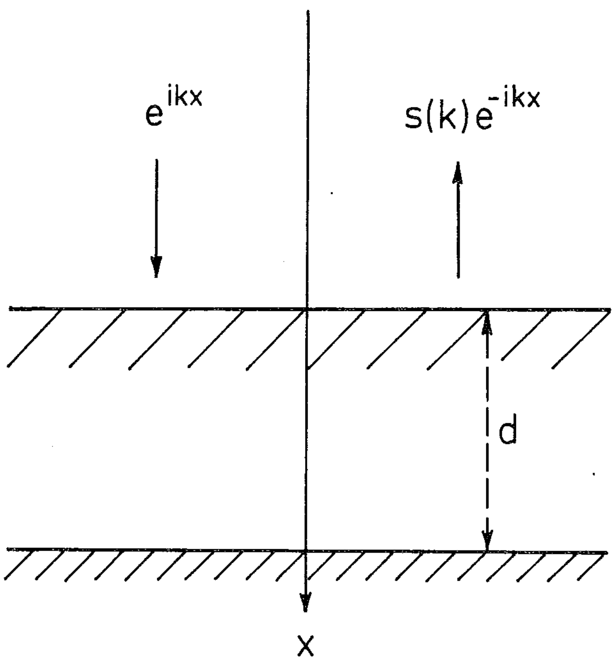

Fig. 2. Reflection from the surface in the presence of a perfectly reflecting surface at given depth.

with

$$V^{\pm}(x, k) = U(x) \pm ikQ(x). \tag{3.2}$$

The auxiliary wave equation (3.1) with the potential V^- does not describe a physical scattering situation and is introduced solely for convenience.

We introduce a pair of fundamental (»Jost») solutions $f^{\pm}(x, k)$ to eqs. (3.1) with the asymptotic behaviour

$$f^{\pm}(x, k) \xrightarrow[x \to -\infty]{} e^{-ikx}. \tag{3.3}$$

These functions satisfy the integral equation (RISKA (1981))

$$f^{\pm}(x, k) = e^{-ikx} + \int_{-\infty}^{x} dx' \frac{\sin k(x-x')}{k} V^{\pm}(x', k) f^{\pm}(x', k) \tag{3.4}$$

in the half space $x \leqslant d$.

The functions $f^+(x, k)$ and $f^-(x, -k)$ then form a pair of linearly independent solution to the eq. (3.1) with the potential $V^+(x, k)$. With the help of this pair we can construct a solution $\psi(x, k)$ with the boundary conditions relevant to the physical scattering situation as

$$\psi(x, k) = f^-(x, -k) + s(k) f^+(x, k). \tag{3.5}$$

Here $s(k)$ is the measurable surface reflection coefficient. Note that in the present case the fundamental solutions attain their asymptotic behaviour (3.3) already at the surface. The reflection coefficient s can therefore be calculated from the electric field at the surface if the primary field is known. Otherwise it may be calculated from the ratio of electric and magnetic fields at the surface by the relation

$$\frac{1 + s(k)}{1 - s(k)} = \frac{1}{c} \frac{E(z=0, k)}{B(z=0, k)}. \tag{3.6}$$

The presence of the perfectly reflecting boundary at $x = d$ demands that

$$\psi(d, k) = 0, \tag{3.7}$$

and thus

$$s(k) = -\frac{f^-(d, -k)}{f^+(d, k)}. \tag{3.8}$$

From this relation and the properties of the fundamental solutions it is readily seen that

$$s^{\star}(k) = s(-k), \tag{3.9}$$

a relation that defines the extrapolation to negative frequencies ($\omega = ck$) for the reflection coefficient. The limit behaviour for the reflection coefficient at low frequencies is

$$\lim_{k \to 0} s(k) = -1, \tag{3.10}$$

and in the case of smooth potential functions at high frequencies (RISKA (1981))

$$\lim_{k \to \infty} s(k) = -e^{2ikd + \int_{-\infty}^{d} dx' Q(x')} \tag{3.11}$$

In order to obtain a convenient integral expression for the reflection coefficient we, following JOST and KOHN (1952), consider the solution $\xi(x, k)$ to eq. (3.1) that has trigonometric asymptotic behaviour and which is defined by the integral equation

$$\xi(x, k) = \sin k(x-d) + \int_{-\infty}^{d} dx' g(x, x') V^{+}(x', k) \xi(x', k) \tag{3.12}$$

with the kernel

$$g(x, x') = \frac{1}{2k} [\sin k(x + x' - 2d) + \sin k |x - x'|]. \tag{3.13}$$

By taking the limit $x \to -\infty$ in eq. (3.12) one finds the asymptotic behaviour

$$\xi(x, k) \xrightarrow[x \to -\infty]{} \sin k(x-d) + \tau(k) \cos k(x-d), \tag{3.14}$$

where the coefficient $\tau(k)$ is given as

$$\tau(k) = \int_{-\infty}^{d} dx' \frac{\sin k(x'-d)}{k} V^{+}(x', k) \xi(x', k). \tag{3.15}$$

The relation between this coefficient $\tau(k)$ and the physical reflection coefficient $s(k)$ may be obtained by comparison of eqs. (3.5) and (3.14):

$$s(k) = -e^{2ikd} \frac{1 - i\tau(k)}{1 + i\tau(k)}, \tag{3.16}$$

or

$$\tau(k) = i \frac{e^{-2ikd} s(k) + 1}{e^{-2ikd} s(k) - 1}. \tag{3.17}$$

The solution to the inversion problem – *i.e.* the determination of the potential $V^+(x,k)$ from the reflection coefficient $s(k)$ may now be obtained by means of iterative solution of the integral equation (3.12) and substitution into eq. (3.15).

Iterative solution of (3.12) and subsequent substitution into (3.15) gives the relation

$$\tau(k) = \frac{1}{k}\int_{-\infty}^{d} dx \sin^2 k(x-d)\; V^+(x,k)$$
$$+\frac{1}{k}\sum_{n=1}^{\infty}\int_{-\infty}^{d} dx \int_{-\infty}^{d} dx_1 \ldots \int_{-\infty}^{d} dx_n \sin k(x-d)\; V^+(x,k) \tag{3.18}$$
$$g(x,x_1)\; V^+(x_1,k)\, g(x_1,x_2) \ldots\; g(x_{n-1},x_n)\; V^+(x_n,k) \sin k(x_n-d).$$

While this expansion is more complicated than the one originally considered by Jost and Kohn (1952) because of the k-dependent nature of the potential it may be inverted by the same method.

One writes formally

$$\tau(k) = \mu F(k) \tag{3.19}$$

$$V^+(x,k) = \sum_{m=1}^{\infty} \mu^m V_m(x,k). \tag{3.20}$$

Substitution of (3.19) and (3.20) into (3.18) and equation of the coefficients of equal powers of μ on both sides of the resulting equations and finally letting μ approach 1 gives

$$\tau(k) = \frac{1}{k}\int_{-\infty}^{d} dx \sin^2 k(x-d)\;\; V_1(x,k), \tag{3.21}$$

and

$$0 = \int_{-\infty}^{d} dx \sin^2 k(x-d)\, V_m(x,k)$$
$$+\sum_{l=2}^{m}\;\sum_{\Sigma\nu_k=m}\int_{-\infty}^{d} dx_1 \ldots dx_l\, G(x_1,x_2,\ldots x_l;k)\; V_{\nu_1}(x_1,k)\ldots V_{\nu_l}(x_l,k). \tag{3.22}$$

The kernel in the last integral is defined as

$$G(x_1,x_2,\ldots x_l;k) = \sin k(x_1-d)\, g(x_1,x_2)\ldots g(x_{l-1},x_l) \sin k(x_l-d). \tag{3.23}$$

The potential component V_1 is obtained by inverting eq. (3.21) and the higher term V_m^+ by successive inversion of the equation (3.22). Finally the complete potential $V^{+}(x, k)$ is calculated from the expression (3.20) with $\mu=1$. The absence of trapped modes makes this solution algorithm particularly well suited to the geophysical inversion problem.

For the first order terms we obtain by inversion of eq. (3.21) the explicit expressions

$$V_1(x, k) = U_1(x) + ikQ_1(x), \tag{3.24}$$

with

$$\begin{aligned} (x-d)\,U_1(x) &= \frac{4}{\pi}\int_0^\infty dk \sin 2k(x-d)\,\frac{d}{dk}\{k\,Re\,\tau(k)\} \\ Q_1(x) &= -\frac{8}{\pi}\int_0^\infty dk \cos 2k(x-d)\,\{Im\,\tau(k) - Im\,\tau(\infty)\}. \end{aligned} \tag{3.25}$$

To obtain these results we use the fact that by (3.17) and (3.11) $\tau(k)$ is purely imaginary in the high frequency limit. The high frequency limit $Im\,\tau(\infty)$ appears as a parameter in the solution, related to $Q_1(x)$ and $Q(x)$ by

$$Im\,\tau(\infty) = \frac{1}{2}\int_{-\infty}^{d} dx\, Q_1(x) = -\frac{1-e^{\int_{-\infty}^{d} dx' Q(x')}}{1+e^{\int_{-\infty}^{d} dx' Q(x')}}. \tag{3.26}$$

Inversion of eq. (3.22) gives the following explicit expressions for the higher order terms:

$$V_m(x, k) = U_m(x) + ikQ_m(x), \tag{3.27}$$

$$\begin{aligned} U_m(x) = \frac{8}{\pi}\sum_{l=2}^{m}\ \sum_{\Sigma\nu_k=m}\ &\int_0^\infty dk \int_{-\infty}^{d} dx_1 \ldots \int_{-\infty}^{d} dx_l \\ &K(x, x_1, \ldots x_l; k)\, Re\,\{V_{\nu_1}(x_1, k) \ldots V_{\nu_l}(x_l, k)\}, \end{aligned} \tag{3.28}$$

$$\begin{aligned} Q_m(x) = \frac{8}{\pi}\sum_{l=2}^{m}\ \sum_{\Sigma\nu_k=m}\ &\int_0^\infty dk \int_{-\infty}^{d} dx_1 \ldots \int_{-\infty}^{d} dx_l \\ &\frac{1}{k} K(x, x_1, \ldots x_l; k)\, Im\,\{V_{\nu_1}(x_1, k) \ldots V_{\nu_l}(x_l, k)\}. \end{aligned} \tag{3.29}$$

Here the kernel function K is defined as

$$K(x, x_1, \ldots x_l; k) = \cos 2k(x-d)\, G(x_1, \ldots x_l; k). \tag{3.30}$$

In order to illustrate the use of these expressions we shall work out the explicit formulae for the second order terms. The results may be written in the form

$$U_2(x) = \int_{-\infty}^{d} dx_1 \int_{-\infty}^{x_1} dx_2 \{\bar{K}(x, x_1, x_2)\, U_1(x_1)\, U_1(x_2) - \bar{L}(x, x_1, x_2)\, Q_1(x_1)\, Q_1(x_2)\}, \tag{3.31}$$

$$Q_2(x) = \int_{-\infty}^{d} dx_1 \int_{-\infty}^{x_1} dx_2\, \bar{K}(x,x_1,x_2)\, [U_1(x_1)\, Q_1(x_2) + U_1(x_2)\, Q_1(x_1)], \tag{3.32}$$

with the kernels

$$\bar{K}(x, x_1, x_2) = \frac{16}{\pi} \int_0^{\infty} dk K(x, x_1, x_2; k), \tag{3.33}$$

$$\bar{L}(x, x_1, x_2) = \frac{16}{\pi} \int_0^{\infty} dk\, k^2 K(x, x_1, x_2; k). \tag{3.34}$$

The kernel functions $\bar{K}$ and $\bar{L}$ have the explicit expressions

$$\bar{K}(x,x_1,x_2) = \begin{cases} 1 & \text{if } x < x_2 < x - x_1 + d, \\ -1 & \text{if } x_1 + x - d < x_2 < x, \\ 0 & \text{otherwise.} \end{cases} \tag{3.35}$$

and

$$\bar{L}(x,x_1,x_2) = \frac{1}{2}\delta'(x - x_2) - \frac{1}{4}\delta'(x - x_1 - x_2 + d) - \frac{1}{4}\delta'(x + x_1 - x_2 - d). \tag{3.36}$$

The convergence conditions of this iterative algorithm for solving the inversion problem were proven and illustrated by JOST and KOHN (1952) for the case of a purely real potential with no trapped modes. In the present case, while there are no trapped modes the potential is complex and linearly frequency dependent. The complexity of the interaction does not affect the convergence conclusions but the frequency dependence may do so. We do not here undertake a systematic investigation of these convergence properties but illustrate the convergence by means of an analytically solvable example. For the example considered the convergence is very rapid. We also note that in the geophysical case only convergence in the topmost layers will be of interest. This method should however not be expected to converge in the geophysically interesting case of a step function discontinuity in the potential at the surface $x = 0$. For this reason we in the following section develop the method in such a way that the surface discontinuity is treated exactly.

Once the potential components $U(x)$ and $Q(x)$ have been determined the physical permeability and conductivity quantities ϵ', μ' and σ must be obtained by inverting the Liouville transform (2.2). The ratio $\sigma(x)/\epsilon'(x)$ is given in terms of $Q(x)$ by eqn. (2.5b). The remaining ratio ϵ'/μ' that may be determined from the real part of the potential $U(x)$ may be obtained by defining

$$\rho(x) = \left(\frac{\epsilon'(x)}{\mu'(x)}\right)^{1/4}, \tag{3.37}$$

and solving the differential equation (2.5a) for $\rho(x)$. The solution is most conveniently obtained by solving the equivalent integral equation (RISKA (1981))

$$\rho(x) = 1 + x\rho'(0) + \int_0^x dx'(x-x')\,U(x')\,\rho(x'). \tag{3.38}$$

Finally the physical scale z is given by

$$z = \theta(x)\int_0^x \frac{dx}{\sqrt{\epsilon'(x)\mu'(x)}} + \theta(-x)x, \tag{3.39}$$

the integration of which requires knowledge of either one of the permeability quantities in addition to the ratios ϵ'/μ' and σ/ϵ'.

4. *Exact treatment of the surface discontinuity*

In the geophysical reflection problem the permeability and conductivity parameters that appear in the wave equation (2.1) commonly change discontinuously from the vacuum (or air) values $\epsilon' = \mu' = 1$, $\sigma = 0$ to values representative of the surface soil layers. Such a discontinuity leads to oscillations in the reflection coefficient that survive in the high frequency limit and may destroy the convergence of the iterative method of solving the inversion problem developed in the previous section. In order to overcome this problem the iterative procedure should be developed starting not from the potential free solution but from a solution to a model potential that incorporates the surface discontinuity. For this purpose we write the potential $V^{\pm}(x, k)$ in eq. (3.1) in the form

$$V^{\pm}(x, k) = V_0^{\pm}(x, k) + v^{\pm}(x, k), \tag{4.1}$$

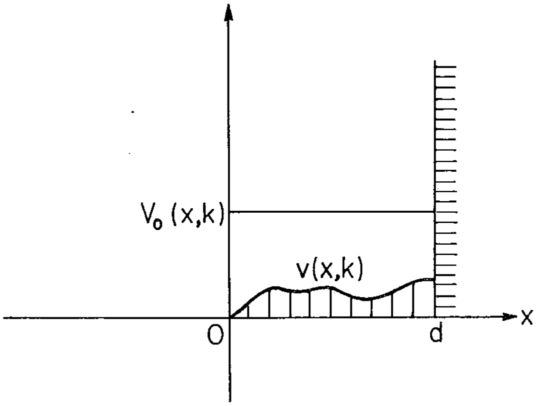

Fig. 3. Separation of the reflecting potential into a model part V_0 representing the surface discontinuity and a residual smooth potential v.

where $V_0^{\pm}$ is the discontinuous model potential and $v^{\pm}$ the smooth residual part of the reflecting potential. In order that the model potential be analytically solvable it may taken to have the form that corresponds to constant permeability, permittivity and conductivity distributions with step discontinuities at the surface.

The differential equation for the model potential is

$$\frac{d^2 y_0^{\pm}(x,k)}{dx^2} + [k^2 - V_0^{\pm}(x,k)]\, y_0^{\pm}(x,k) = 0. \tag{4.2}$$

The corresponding fundamental solutions $f_0^{\pm}(x,k)$ are given by the integral equation (3.4):

$$f_0^{\pm}(x,k) = e^{-ikx} + \int_{-\infty}^{x} dx' \frac{\sin k(x-x')}{k} V_0^{\pm}(x';k)\, f_0^{\pm}(x';k). \tag{4.3}$$

The explicit expressions are given in Appendix 2. We also define the regular solution φ_0 for the model problem (4.2) by the integral equation

$$\varphi_0(x,k) = \frac{\sin k(x-d)}{k} - \int_{x}^{d} dx' \frac{\sin k(x-x')}{k} V_0^{+}(x',k)\, \varphi_0(x',k). \tag{4.4}$$

The solution $\xi_0(x,k)$ to eq. (4.2) that has trigonometric behaviour at $x \to -\infty$ is given by the integral equation (3.12) as

$$\xi_0(x,k) = \sin k(x-d) + \int_{-\infty}^{d} dx' g(x,x')\, V_0^{+}(x',k)\, \xi_0(x',k). \tag{4.5}$$

with the Green's function $g(x,x')$ defined in eq. (3.13).

Consider now the function $\xi(x,k)$ defined by the integral equation

$$\xi(x,k) = \xi_0(x,k) + \int_{-\infty}^{d} dx'\, G(x,x')\, v^+(x',k)\, \xi(x',k) \tag{4.6}$$

with the kernel

$$G(x,x') = \theta(x-x')\,\varphi_0(x)\chi_0(x') + \theta(x'-x)\,\chi_0(x)\,\varphi_0(x'). \tag{4.7}$$

Here φ_0 is the regular solution for the model problem (4.4) and χ_0 is the corresponding irregular solution defined as

$$\chi_0(x,k) = \frac{1}{2ik}\{f_0^{-\prime}(d,-k)\, f_0^+(x,k) - f_0^{+\prime}(d,k)\, f_0^-(x,-k)\}. \tag{4.8}$$

This solution satisfies the integral equation (4.4) provided the inhomogeneous term is replaced by $\cos k(x-d)$. Noting that $\chi_0(d,k) = 1$, $\chi_0'(d,k) = 0$ and that hence

$$\varphi_0'(x,k)\,\chi_0(x,k) - \chi_0'(x,k)\,\varphi_0(x,k) = 1 \tag{4.9}$$

it is readily seen that $\xi(x,k)$ is a solution to the fundamental differential equation (3.1) with the complete potential (4.1) that reduces to the model solution ξ_0 (4.5) in the limit $v^+ \to 0$.

To obtain the asymptotic behaviour of the solution $\xi(x,k)$ in the limit $x \to -\infty$ we note that by (3.14)

$$\xi_0(x,k) \to \sin k(x-d) + \tau_0(k)\cos k(x-d) \tag{4.10}$$

where $\tau_0(k)$ is a model problem reflection coefficient defined by

$$\tau_0(k) = \int_{-\infty}^{d} dx'\, \frac{\sin k(x'-d)}{k}\, V_0^+(x',k)\, \xi_0(x',k). \tag{4.11}$$

This coefficient is by eqs. (3.8) and (3.16) related to the model problem surface reflection coefficient $s_0(k)$ by

$$s_0(k) = -e^{2ikd}\, \frac{1-i\tau_0(k)}{1+i\tau_0(k)}. \tag{4.12}$$

From the definitions (4.7) and (4.8) the asymptotic behaviour of the Green's function (4.7) is found to be

$$G(x, x') \xrightarrow[x\to-\infty]{} Z[\cos k(x-d) + \mu_0 \sin k(x-d)]\,\varphi_0(x'), \tag{4.13}$$

with the coefficients μ_0 and Z defined as

$$\mu_0 = -i\,\frac{f_0^{-\prime}(d,-k) + e^{2ikd} f_0^{+\prime}(d,k)}{f_0^{-\prime}(d,-k) - e^{2ikd} f_0^{+\prime}(d,k)}, \tag{4.14}$$

$$Z = \frac{e^{-ikd}}{2ik}\,[f_0^{-\prime}(d,-k) - e^{2ikd} f_0^{+\prime}(d,k)]. \tag{4.15}$$

Collecting the results we obtain

$$\xi(x,k) \xrightarrow[x\to-\infty]{} (1+\mu_0\tau')\Big\{\sin k(x-d) + \frac{\tau_0+\tau'}{1+\mu_0\tau'}\cos k(x-d)\Big\}, \tag{4.16}$$

with the coefficient τ' defined as

$$\tau' = Z\int_{-\infty}^{d} dx'\,\varphi_0(x',k)\,v^+(x',k)\,\xi(x',k). \tag{4.17}$$

By comparing the asymptotic behaviour (4.16) to that of the physical solution (3.6) one obtains the following expression for the reflection coefficient $s(k)$:

$$s(k) = -e^{2ikd}\,\frac{1 - i\,\dfrac{\tau_0+\tau'}{1+\mu_0\tau'}}{1 + i\,\dfrac{\tau_0+\tau'}{1+\mu_0\tau'}} \tag{4.18}$$

In the limit $v^+\to 0$ the coefficient τ' vanishes and hence the reflection coefficient $s(k)$ reduces to that for the model problem (4.12). The model problem reflection coefficient contains all the oscillations caused by the surface discontinuity. Since $s_0(k)$ and $\mu_0(k)$ are known analytically (see Appendix 2) the coefficient τ' may be calculated from the reflection coefficient $s(k)$ using (4.18) as

$$\tau'(k) = -\frac{1 + e^{-2ikd}s(k) - i\tau_0(1 - e^{2ikd}s(k))}{\mu_0(1 + e^{-2ikd}s(k)) - i(1 - e^{-2ikd}s(k))}. \tag{4.19}$$

The inversion problem now consists of determining the residual potential v^+ from the coefficient $\tau'(k)$. The solution may be constructed by an iterative method similar to that develop in the previous section.

This method is based on iterative solution of the integral equation (4.6) and successive substitution into equation (4.17) for $\tau'(k)$. The procedure yields the following series expansion for $\tau'(k)$:

$$\tau'(k) = Z\int_{-\infty}^{d} dx\, \varphi_0(x,k)\, \xi_0(x,k)\, \nu^+(x,k)$$
$$+ Z\sum_{n=1}^{\infty} \int_{-\infty}^{d} dx \int_{-\infty}^{d} dx_1 \dots \int_{-\infty}^{d} dx_n\, \varphi_0(x,k)\, \nu^+(x,k) \tag{4.20}$$
$$G(x,x_1)\, \nu^+(x_1,k) \dots G(x_{n-1},x_n)\, \nu^+(x_n,k)\, \xi_0(x_n,k).$$

As in eqs. (3.18) we make the ansätze

$$\frac{\tau'(k)}{Z} = \mu \tilde{F}(k) \tag{4.21}$$

$$\nu^+(x,k) = \sum_{m=1}^{\infty} \mu^m \nu_m^+(x,k). \tag{4.22}$$

Substitution of these expressions into eq. (4.20) and equating terms with equal powers of μ then after setting $\mu = 1$ gives

$$\tau'(k) = Z\int_{-\infty}^{d} dx\, \varphi_0(x,k)\, \xi_0(x,k)\, \nu_1^+(x,k) \tag{4.23}$$

and

$$\int_{-\infty}^{d} dx\, \varphi_0(x,k)\, \xi_0(x,k)\, \nu_m(x,k)$$
$$+ \sum_{l=2}^{m} \sum_{\Sigma\nu_k = m} \int_{-\infty}^{d} dx_1 \dots dx_l\, \tilde{G}(x_1,\dots x_l;k)\, \nu_{\nu_1}^+(x_1,k) \dots \nu_{\nu_l}^+(x_l,k) = 0. \tag{4.24}$$

The kernel $\tilde{G}$ in (4.24) is defined as

$$\tilde{G}(x_1,\dots x_l;k) = \varphi_0(x_1,k)\, G(x_1,x_2) \dots G(x_{l-1},x_l)\, \xi_0(x_l,k). \tag{4.25}$$

The potential component ν_1^+ may be obtained by numerical inversion of eq. (4.23) and the higher components by numerical inversion of eq. (4.24) for successive values of m. The complete potential is finally constructed as the sum (4.22) with $\mu = 1$. In practice this algorithm that treats the surface discontinuity exactly should not be more cumbersome than the method developed in section 3. The explicit expressions for the model problem solutions needed in the algorithm are given in Appendix 2.

5. *Discussion*

The ultimate value of the inverse scattering solution will depend on the accuracy of the solution in the topmost layers. All inverse scattering solutions are sensitive to the extrapolation of the measured reflection coefficient into the unmeasurable high frequency region. Thus both an »exact» solution of the Jaulent integral equations and an iterative solution obtained by the present method should be expected to be completely unreliable at large depths into which the radiation penetrates poorly. Thus solving the inverse scattering integral equations will not in practice yield more information of practical utility than an iterative method that is reliable only in the topmost layers. The iterative method does however bring the great practical advantage of reducing the problem to quadrature.

A priori it might appear that the introduction of an artificial perfectly conducting layer at given depth should introduce a large measure of nonuniqueness in the inversion problem solution. The dependence of the depth of this layer however simply represents a certain choice of high frequency extrapolation of the reflection coefficient as may be seen from eq. (3.11). As such an extrapolation is in any case necessary in practice the introduction of the reflecting layer actually serves to only simplify the problem.

In the common situation of a discontinuous change of the conductivity and permeability parameters at the surface of the earth the iterative method will converge well only if the surface discontinuity is treated exactly. The formalism appropriate to this situation is developed in section 4. Numerically the method that treats the surface discontinuity exactly should not be far more cumbersome than the general Jost-Kohn type method developed in section 3.

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Appendix 1

An analytically solvable example

The inverse scattering problem may be solved exactly for the reflection coefficient (Riska, 1982)

$$s(k) = -\frac{b}{a}\frac{k+ia}{k+ib}e^{2ikd}, \quad a > b > 0. \tag{A1.1}$$

The solutions for the potential components U and Q are then

$$U(x) = -\frac{a(a-b)}{b}\frac{e^{-2a(d-x)}}{1+\frac{a-b}{b}e^{-2a(d-x)}}\left\{2a - \frac{3a(a-b)}{b}\frac{e^{-2a(d-x)}}{\left(1+\frac{a-b}{b}e^{-2a(d-x)}\right)}\right\},$$

$$Q(x) = -2\frac{a(a-b)}{b}\frac{e^{-2a(d-x)}}{1+\frac{a-b}{b}e^{-2a(d-x)}} \tag{A1.2}$$

It is instructive to compare the potential components calculated by the iterative algorithm to these expressions. From (A1.1) and (3.17) we obtain the coefficient τ as

$$\tau(k) = -i\frac{(a^2-b^2)k^2 - 2iab(a-b)k}{(a+b)^2k^2 + 4a^2b^2}. \tag{A1.3}$$

Using this expression one obtains the first order potential components U_1 and Q_1 from Eq. (3.25) as

$$U_1(x) = -\frac{16a^2b^2(a-b)}{(a+b)^3}e^{-2\xi(d-x)},$$

$$Q_1(x) = -\frac{8ab(a-b)}{(a+b)^2}e^{-2\xi(d-x)}. \tag{A1.4}$$

Here the coefficient ξ has been defined as

$$\xi = \frac{2ab}{a+b}. \tag{A1.5}$$

It is readily seen that to first order in the parameter $a-b$ the functions U_1 and Q_1 agree with the exact results (A1.2).

The second and third order corrections to the potential components as obtained from eqs. (3.28)–(3.34) are

$$
\begin{aligned}
U_2(x) &= \;\; 16\,\frac{a^2b^2(a-b)^2}{(a+b)^4}\,\{-3 + 2\xi(d-x) + 5\,e^{-2\xi(d-x)}\}\,e^{-2\xi(d-x)}\,,\\
Q_2(x) &= -16\,\frac{ab(a-b)^2}{(a+b)^3}\,\{1 - \xi(d-x) - e^{-2\xi(d-x)}\}\,e^{-2\xi(d-x)}\,,\\
U_3(x) &= -16\,\frac{a^2b^2(a-b)^3}{(a+b)^5}\,\{2\xi^2(d-x)^2 - 8\xi(d-x) + 6 + [20\xi(d-x) - 20]\,e^{-2\xi(d-x)}\\
&\quad + 16\,e^{-4\xi(d-x)}\}\,e^{-2\xi(d-x)}\,,\\
Q_3(x) &= -8\,\frac{ab(a-b)^3}{(a+b)^4}\,\{2\xi^2(d-x)^2 - 6\xi(d-x) + 3 + (8\xi(d-x) - 6)\,e^{-2\xi(d-x)}\\
&\quad + 4\,e^{-4\xi(d-x)}\}\,e^{-2\xi(d-x)}.
\end{aligned}
\tag{A1.6}
$$

Adding these functions to the corresponding first order terms in (A1.4) one finds that the results agree with the exact potential components (A1.2) to third order in the quantity $a-b$. The parameter $a-b$ measures the deviation from unitarity of the reflection coefficient. The convergence of the iterative algorithm thus depends on the closeness to unitary of the reflection coefficient, an observation that is already implicit in Jaulent's work (1976). It is worth emphasizing that in the geophysical case the reflection coefficient is usually very close to unitary. In the example above it turns out numerically that the sum of the first and second order terms approximate the exact potential fairly well for a wide choice of a and b.

Appendix 2

Explicit formulae for the model problem

We choose the model potential $V_0^{\pm}$ so that it describes a step discontinuity in the permeability and conductivity parameters at the surface ($x = z = 0$):

$$\epsilon = \epsilon_0[\theta(-x) + \epsilon_0'\theta(x)], \tag{A2.1a}$$

$$\mu = \mu_0[\theta(-x) + \mu_0'\theta(x)], \tag{A2.1b}$$

$$\sigma = \sigma_0\theta(x). \tag{A2.1c}$$

The corresponding potential $V_0^{\pm}$ is then according to eqs. (2.5)

$$V_0^{\pm}(x,k) = (\alpha-1)\,\delta'(x)\left[\frac{\theta(x)}{\alpha} + \theta(-x)\right] \mp i\,\frac{k\sigma_0}{c\epsilon_0\epsilon_0'}\,\theta(x)' \tag{A2.2}$$

with α defined as

$$\alpha = \left(\frac{\epsilon_0'}{\mu_0'}\right)^{1/4}. \tag{A2.3}$$

The solutions to the differential equation (4.2) with the potential (A2.2) can be constructed with the help of the boundary conditions at $x = 0$:

$$f(0-, k) = \frac{1}{\alpha} f(0+, k), \tag{A2.4a}$$

$$f'(0-, k) = \alpha f'(0+, k). \tag{A2.4b}$$

These are the electromagnetic boundary conditions for the transformed problem (2.3).

The Jost functions $f_0^{\pm}$ (4.3) for the model problem are

$$f_0^+(x, k) = \theta(-x)\, e^{-ikx} + \frac{\alpha}{2}\,\theta(x) \left[\left(1 + \frac{1}{\delta\alpha^2}\right) e^{-ik'x} + \left(1 - \frac{1}{\delta\alpha^2}\right) e^{ik'x}\right], \tag{A2.5a}$$

$$f_0^-(x, -k) = \theta(-x)\, e^{ikx} + \frac{\alpha}{2}\,\theta(x) \left[\left(1 + \frac{1}{\delta\alpha^2}\right) e^{ik'x} + \left(1 - \frac{1}{\delta\alpha^2}\right) e^{-ik'x}\right]. \tag{A2.5b}$$

Here we have used the notation

$$k' = k\left\{\sqrt{\frac{\sqrt{1 + \sigma_0^2/c^2\epsilon_0^2\epsilon_0'^2k^2} + 1}{2}} + i\sqrt{\frac{\sqrt{1 + \sigma_0^2/c^2\epsilon_0^2\epsilon_0'^2k^2} - 1}{2}}\right\}, \tag{A2.6}$$

and

$$\delta = \frac{k'}{k}. \tag{A2.7}$$

The Jost functions (A2.5) lead to the following expression for the model problem reflection coefficient $s_0(k)$ by the definition (3.8):

$$s_0(k) = -e^{2ik'd}\, \frac{\left(1 + \frac{1}{\delta\alpha^2}\right) + \left(1 - \frac{1}{\delta\alpha^2}\right) e^{-2ik'd}}{\left(1 + \frac{1}{\delta\alpha^2}\right) + \left(1 - \frac{1}{\delta\alpha^2}\right) e^{2ik'd}}. \tag{A2.8}$$

Here d is the depth (in the transformed variable x) of the perfectly conducting layer.

The auxiliary coefficients $\tau_0(k)$, $\mu_0(k)$ and Z for the model problem defined in eqs. (4.11), (4.14) and (4.15) are then

$$\tau_0(k) = -i\,\frac{\left(1+\frac{1}{\delta\alpha^2}\right)\left[1-e^{2i(k'-k)d}\right]+\left(1-\frac{1}{\delta\alpha^2}\right)\left[e^{2ik'd}-e^{-2ikd}\right]}{\left(1+\frac{1}{\delta\alpha^2}\right)\left[1+e^{2i(k'-k)d}\right]+\left(1-\frac{1}{\delta\alpha^2}\right)\left[e^{2ik'd}+e^{-2ikd}\right]}, \quad \text{(A2.9a)}$$

$$\mu_0(k) = -i\,\frac{\left(1+\frac{1}{\delta\alpha^2}\right)\left[1-e^{-2i(k'-k)d}\right]-\left(1-\frac{1}{\delta\alpha^2}\right)\left[e^{-2ik'd}-e^{2ikd}\right]}{\left(1+\frac{1}{\delta\alpha^2}\right)\left[1+e^{-2i(k'-k)d}\right]-\left(1-\frac{1}{\delta\alpha^2}\right)\left[e^{-2ik'd}+e^{2ikd}\right]}, \quad \text{(A2.9b)}$$

$$Z = \frac{\delta}{4}\, e^{i(k'-k)d}\left\{\left(1+\frac{1}{\delta\alpha^2}\right)\left(1+e^{-2i(k'-k)d}\right)-\left(1-\frac{1}{\delta\alpha^2}\right)\left(e^{-2ik'd}+e^{2ikd}\right)\right\}. \quad \text{(A2.9c)}$$

Finally the regular solution $\varphi_0(x, k)$ (4.4) for the model problem is

$$\varphi_0(x,k) = \frac{\alpha}{2k}\,\theta(-x)\left\{\left(1+\frac{1}{\delta\alpha^2}\right)\sin(kx-k'd)+\left(1-\frac{1}{\delta\alpha^2}\right)\sin(kx+k'd)\right\} + \theta(x)\,\frac{\sin k'(x-d)}{k\delta}. \quad \text{(A2.10)}$$

and the corresponding solution $\xi_0(x, k)$ (4.5):

$$\xi_0(x,k) = \frac{4k}{\alpha}\,\frac{e^{i(k'd-kd)}}{\left(1+\frac{1}{\delta\alpha^2}\right)\left[1+e^{2i(k'-k)d}\right]+\left(1-\frac{1}{\delta\alpha^2}\right)\left[e^{2ik'd}+e^{-2ikd}\right]}\,\varphi_0(x,k). \quad \text{(A2.11)}$$

Because of the discontinuity in the model problem potential $V_0^{\pm}$ defined in eq. (A2.2) the solution of the differential equation (2.5a) in terms of the integral (3.38) must be modified. For this purpose we write the function $\rho(x)$ (3.37) as

$$\rho(x) = \rho_0(x) + \tilde{\rho}(x) \quad \text{(A2.12)}$$

where $\rho_0(x)$ is the solution for the model problem and $\tilde{\rho}(x)$ the difference between the actual and the model solution. The explicit expression for ρ_0 is

$$\rho_0(x) = \alpha\theta(x) + \theta(-x). \quad \text{(A2.13)}$$

Using the properties of this function one may derive the following integral equation for $\tilde{\rho}(x)$:

$$\tilde{\rho}(x) = x\tilde{\rho}'(0) + \int_0^x dx'(x-x')\,\mathrm{Re}\,v^{+}(x')\,[\alpha+\tilde{\rho}(x')], \quad \text{(A2.14)}$$

which is well behaved and solvable by iteration. The equations (A2.12) and (A2.14) thus replace eq. (3.38) in the case of using the model potential solution.