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ELECTROMAGNETIC INDUCTION IN THE EARTH BY A PLANE WAVE OR BY FIELDS OF LINE CURRENTS HARMONIC IN TIME AND SPACE

by

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Abstract

Studies of geomagnetic disturbances and connected electric fields, *i.e.* of electromagnetic induction in the earth, yield information on the structure of the earth and on the ionospheric and magnetospheric current systems. In this paper three theoretical models of the primary field of the induction are formulated:

1. a harmonic plane wave propagating vertically downwards
2. the field caused by an infinitely long horizontal straight line current oscillating harmonically in time
3. the field caused by a similar line current which, in addition, has a longitudinal harmonic space dependence, implying an accumulation of charge on the line.

The earth is assumed to be an infinite half-space with a flat surface. The electromagnetic properties of the earth are assumed to be laterally constant and piecewise constant in the vertical direction. Formal extensions to arbitrary vertical variations are included and so, too, in the third model, are lateral variations in the transverse horizontal direction. In the mathematical calculations, however, it proved necessary, to assume that the lateral variation vanishes at the earth's surface.

Formulae for the total disturbance field on the earth's surface are derived, basing rigorously on Maxwell's equations and on electromagnetic boundary conditions. The final formulae for the last two models are complicated integrals over a horizontal wave number.

The first two models are included in order to give complete and non-approximative treatments of them. Comments on previous publications dealing with the first two models are presented, although the main results derived earlier are good approximations of rigorous formulae.

The third model, whose treatment is similar to the second, is an extension to the theory of induction published earlier. The longitudinal space dependence implies the existence of non-zero electric-field components perpendicular to the current, and of a parallel non-zero magnetic-field component. The possibility of assigning longitudinal attenuation to the primary source is also investigated and shown to be unacceptable.

The general and rigorous treatment in this work may make it applicable also to other problems of electromagnetism.

1. *Introduction*

Time variations of the electric current system in the earth's ionosphere and magnetosphere are caused mainly by the sun in complex ways that are not yet fully understood. These variations lead to fluctuations in the earth's magnetic field (the geomagnetic field). Irregular fluctuations are generally known as geomagnetic disturbances, and the most violent of them are called geomagnetic storms.

According to basic electromagnetic theory, an electric field is inevitably connected with fluctuations in the geomagnetic field (Faraday's law of induction). The magnetic variations and the electric field are further affected by secondary currents and charges induced in the earth, in addition to their primary source currents and charges in the ionosphere and in the magnetosphere. According to HULTQVIST, 1973, about 60 % of the horizontal disturbance component of the geomagnetic field at the earth's surface is caused by primary sources, and 40 % by secondary sources. The whole phenomenon can be called »electromagnetic induction in the earth». The term »geomagnetic induction» is also used. The study of geomagnetic variations and associated electric fields is known as magnetotellurics.

The main geomagnetic field, which originates almost entirely inside the earth, and upon which the fluctuations are superimposed, is approximately a dipole field, with a magnitude in Finland of about 50,000 nT. The largest time variations in the geomagnetic field are a few thousand nT. A typical changing time of the main field is several years, so it can be considered static, and without any electric field. The main geomagnetic field can be totally ignored in the treatment of electromagnetic induction in the earth. Thus »field» in this paper always refers to a variation in the geomagnetic field or to the connected electric field, or both. Real fluctuations in the geomagnetic field are discussed from the standpoint of magnetotellurics by KAUFMAN and KELLER, 1981, pp. 1–38.

If all the sources of the electromagnetic field situated outside the earth and the electromagnetic properties within the earth were known, the whole phenomenon could in principle be solved theoretically using Maxwell's equations and boundary conditions. But the primary sources and the structure of the earth are not exactly known. In fact they are so complicated that precise calculations would be impossible in practice. So only approximative and simplified models can be used in theoretical treatments. The fitting of the observed electromagnetic field to such models gives information on the ionospheric and magnetospheric currents and charges, and on the structure of the earth.

This purely theoretical work deals with three models for the primary field caused by sources in the earth's ionosphere and magnetosphere. In Chapter 2 the field is a harmonic plane wave propagating vertically downwards, and corresponding to an infinite horizontal current sheet as the primary source. The primary field in Chapter 3 is created by an infinitely long straight line current oscillating harmonically in time, and situated horizontally above the earth's surface. In Chapter 4 a sinusoidal space dependence along the line is added to the source in Chapter 3; *i.e.* the current has the form of a wave propagating along the line. The current in Chapter 4 is accompanied by a primary charge.

The models presented in Chapters 2 and 3 have been discussed in the literature. For Chapter 2 CAGNIARD, 1953, WAIT, 1962, and KAUFMAN and KELLER, 1981, pp. 39–74, can be referred to. The induction caused by the primary source in Chapter 3 has been treated by PRICE, 1950, pp. 398–404, LAW and FANNIN, 1961, ALBERTSON and VAN BAELEN, 1970, KELLER and FRISCHKNECHT, 1970, pp. 306–307, PARK, 1973 and 1974, and KAUFMAN and KELLER, 1981, pp. 113–126. Chapters 2 and 3 are nevertheless included in this paper to give a complete and thorough treatment of these models as a background to Chapter 4. The analyses in Chapters 2 and 3 are based directly on classical electromagnetic theory and should therefore be readable and understandable without the above-mentioned references, though comments on these earlier publications are included. The present discussion also contains formulae derived rigorously from Maxwell's equations and from the electromagnetic boundary conditions. Hence displacement currents are not neglected though, owing to the low frequencies, such neglect would evidently be acceptable in dealing with the theory of geomagnetic induction in the earth. Thus the applicability of this work is not necessarily confined to electromagnetic induction in the earth.

The model in Chapter 4 is a generalization of that presented in Chapter 3, and has not been discussed earlier in the literature. Chapter 4 likewise involves a rigorous application of Maxwell's equations and boundary conditions.

In this work the earth is assumed to be a half-space, making the earth's surface an infinite plane. The air is initially regarded as electromagnetically free space with vacuum permittivity and permeability and zero conductivity. To simplify the mathematical treatment, however, a very small positive conductivity is assigned to the air in Chapters 3 and 4.

The first model for the structure of the earth discussed in Chapters 2, 3 and 4 consists of horizontal layers, thus resembling practical situations. Each layer has a constant conductivity, a constant permittivity and a constant permeability, all of which are scalars implying isotropy. Later, generalizations are made in which the parameters, though still scalars, are allowed to be arbitrary functions of depth. In Chapter 4 lateral variations in the transverse horizontal direction are also formally involved, though in the course of the treatment this horizontal dependence is assumed to vanish totally at the earth's surface so as to make the mathematical treatment much easier. The special case of a homogeneous earth is discussed separately in Chapters 2, 3 and 4.

The validity of the treatments in these chapters is discussed in Chapter 5.

The horizontal-layers model is included in the treatment of arbitrary vertical variations. So it would seem logical to deal first, in each chapter, with the most general model and then discuss the horizontally layered earth as a special case. But the latter model permits an explicit solution in terms of the number, thicknesses and electromagnetic parameters of the layers of the earth. On the other hand, no explicit solution is possible in the case of arbitrary variations. By changing the parameters describing the horizontally layered model, which may have any values, a broad variety of earth structures can be obtained. These considerations dictate the order of treatment.

Lateral variations of the structure of the earth, which always complicate the solution of geomagnetic induction problems, are discussed by Jones and Price, 1971, Hobbs, 1975, and Kaufman and Keller, 1981, pp. 175–404, among others.

The principle of treatment is the same in each of Chapters 2, 3 and 4. The expressions for the primary electromagnetic field are given first. The secondary field created in the air by the earth and the field inside the earth are then calculated with Maxwell's differential equations, and boundary conditions are used to connect the unknown integration constants to each other and to the known quantities. The aim in each chapter is to derive formulae for the disturbance field on the earth's surface. This field, which is the sum of the primary and the secondary electromagnetic fields, corresponds to actual observations.

It could be thought that the secondary field produced by the earth might influence the primary sources, thus altering the primary field. But it is assumed throughout this work that the primary fields are the actual fields originating outside the earth, so they already include all possible effects of the secondary fields. Ducruix *et al.*, 1977, make a similar assumption in their paper on geomagnetic induction. Even without such an assumption it is clear that the effect of a primary field on its own sources is already included in the expressions of the primary sources and field. If the air is not taken to be electromagnetically free space, the primary and secondary fields produce additional induced sources in the air. In this wotk the effect of the former, but not that of the latter, is contained in the primary field.

It is assumed throughout that the functions are sufficiently well-behaved to permit the mathematical procedures. Often in this paper unknown quantities are solved from sets of linear equations. In these cases it is always assumed that a single-valued finite solution exists, which mathematically means to say that the determinant of the set of equations differs from zero.

Certain other mathematical subjects and formulae utilized are presented in Appendix A. Elements of classical electromagnetic theory upon which the treatments in Chapters 2, 3 and 4 are based are given in Appendix B. The matters discussed in Appendices A and B can also be found in the literature. In this paper, however, they are presented to make Chapters 2, 3 and 4 easier to read. Appendix C contains a discussion of cylindrical electromagnetic fields, which is necessary for Chapter 4. Surface waves, a topic related to Chapter 4, are also discussed briefly in Appendix C.

As stated above, the study of electromagnetic induction in the earth is useful for investigating the structure of the earth and phenomena in the ionosphere and magnetosphere. It also has another practical application: induction in earthed conductors. The electric field associated with geomagnetic disturbances appears as potential differences between separate points on the earth's surface. Such a voltage creates an electric current in any conductor that is earthed at different points. In oil and gas pipelines, for example, such currents may cause corrosion problems (CAMPBELL, 1978 and 1980). In power transmission lines they cause saturation of transformers, which may lead to disturbances in the operation of power transmission systems and can even damage the transformers (ALBERTSON *et al.*, 1973, TAKALA, 1979 and 1980, PIRJOLA, 1980).

In the theoretical calculation of currents induced in conductors, potential differences between the earthing points if there was no conductor have to be calculated. For this purpose expressions of potential differences between two arbitrary points on the earth's surface are also presented in this paper. As the magnetic field is not time-independent, a potential difference is not single-valued, but depends on the path along which the electric field is integrated. Obviously it should always be integrated along the conductor. In this work, however, all the potential differences are calculated along the shortest path on the earth's surface, *i.e.* a straight line. The use of these potential differences, say in calculations on straight power lines, causes an error which is clearly negligible, because the height of power lines is very small compared to the wavelength in free space and to the depth of penetration in the earth associated with significant frequencies in geomagnetic variations.

2. *Induction in the case of a plane wave primary field*

2.1. Description of the model and the expressions of the primary field

We shall describe the earth as a half-space. Thus the earth's surface is an infinite plane. The other half-space, the air, is regarded as electromagnetically free space having the permittivity ϵ_0, the permeability μ_0 and zero conductivity. Electromagnetically, the air in the vicinity of the earth's surface behaves almost as free space. Assume further that the primary electromagnetic field originating from ionospheric and magnetospheric sources is a plane wave having a harmonic time-dependence $e^{i\omega t}$ with $\omega > 0$ and propagating downwards perpendicularly to the earth's surface. Such a field is created by an infinite horizontal current sheet above the earth.

Using the standard Cartesian coordinate system, where the xy-plane coincides with the earth's surface and the x-axis points northwards, the y-axis eastwards and the z-axis downwards, normally used in geomagnetic considerations, the primary fields have, according to Section B.6, the expressions

$$\bar{E}_i = \bar{E}_0^+ e^{i(\omega t - k_0 z)} \tag{2.1}$$

and

$$\bar{B}_i = \bar{B}_0^+ e^{i(\omega t - k_0 z)} \tag{2.2}$$

where

$$\bar{E}_0^+ = E_{0x}^+ \hat{e}_x + E_{0y}^+ \hat{e}_y = c\bar{B}_0^+ \times \hat{e}_z \tag{2.3}$$

and

$$\bar{B}_0^+ = B_{0x}^+ \hat{e}_x + B_{0y}^+ \hat{e}_y \, . \tag{2.4}$$

Here k_0 is the real and positive propagation constant of free space given by formula (B.91), c is the speed of light in free space and the unit vectors $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$ point in the directions of the positive x-, y- and z-axes, respectively. (See also the end of Section B.9.)

2.2. Induction in a horizontally layered earth

Assume that the earth, *i.e.* the lower half-space, can be divided into horizontal layers each of which has constant conductivity σ_j ($\neq\infty$), constant permittivity ϵ_j

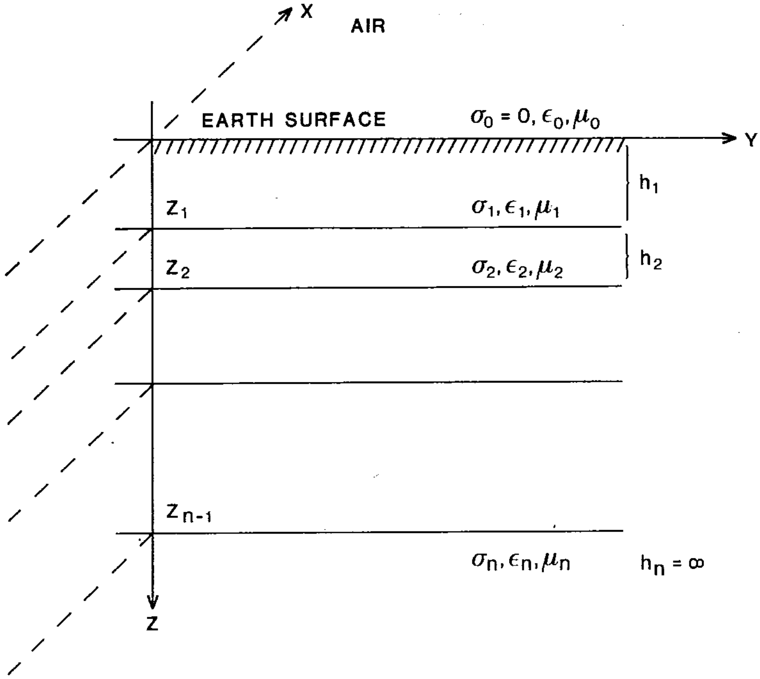

Fig. 1. Model of the earth consisting of horizontal layers. This model is used in Chapters 2, 3 and 4.

and constant permeability μ_j (Fig. 1). The thicknesses of the layers are $h_1, h_2, \ldots, h_{n-1}$ and $h_n (= \infty)$.

Because of symmetry it is natural that the only space coordinate upon which the electromagnetic fields appearing in this problem depend is z. Due to the boundary conditions (Section B.7) it also seems natural to assume that all fields are harmonic with the same angular frequency ω (see FEYNMAN *et al.,* 1964, p. 33–8), *i.e.* the time-dependence is expressed by $e^{i\omega t}$. The earth causes a secondary field in the air, which propagates upwards and may be called reflected. This electromagnetic field is

$$\bar{E}_r = \bar{E}_0^- e^{i(\omega t + k_0 z)} \tag{2.5}$$

and

$$\bar{B}_r = \bar{B}_0^- e^{i(\omega t + k_0 z)} \tag{2.6}$$

where

$$\bar{E}_0^- = E_{0x}^- \hat{e}_x + E_{0y}^- \hat{e}_y = -c\bar{B}_0^- \times \hat{e}_z \tag{2.7}$$

and

$$\bar{B}_0^- = B_{0x}^- \hat{e}_x + B_{0y}^- \hat{e}_y \tag{2.8}$$

(see Sections B.6 and B.9).

The electromagnetic field appearing within the earth is also obtained from the discussions in Section B.6 and at the end of B.9. In all but the lowest layer the field consists of two parts, one propagating downwards, the other propagating upwards. Thus in the j^{th} layer ($j = 1,\dots,n-1$)

$$\bar{E} = \bar{E}_j^+ e^{i(\omega t - k_j z)} + \bar{E}_j^- e^{i(\omega t + k_j z)} \tag{2.9}$$

and

$$\bar{B} = \bar{B}_j^+ e^{i(\omega t - k_j z)} + \bar{B}_j^- e^{i(\omega t + k_j z)} \tag{2.10}$$

where k_j is the propagation constant of the j^{th} layer given by the formulae (B.41) and (B.42) and

$$\bar{E}_j^+ = E_{jx}^+ \hat{e}_x + E_{jy}^+ \hat{e}_y = \frac{\omega}{k_j} \bar{B}_j^+ \times \hat{e}_z\,, \tag{2.11}$$

$$\bar{B}_j^+ = B_{jx}^+ \hat{e}_x + B_{jy}^+ \hat{e}_y\,, \tag{2.12}$$

$$\bar{E}_j^- = E_{jx}^- \hat{e}_x + E_{jy}^- \hat{e}_y = -\frac{\omega}{k_j} \bar{B}_j^- \times \hat{e}_z \tag{2.13}$$

and

$$\bar{B}_j^- = B_{jx}^- \hat{e}_x + B_{jy}^- \hat{e}_y\,. \tag{2.14}$$

It is clear that the electromagnetic field in the lowest layer consists only of a downward propagating wave, which for $\sigma_n \neq 0$ is also attenuated downwards, because the layer continues homogeneously to infinity (see *e.g.* STRATTON, 1941, p. 511). The upward propagating wave would grow to infinity as z increases, if σ_n differs from zero. Thus in the lowest layer

$$\bar{E} = \bar{E}_n^+ e^{i(\omega t - k_n z)} \tag{2.15}$$

and

$$\bar{B} = \bar{B}_n^+ e^{i(\omega t - k_n z)} . \tag{2.16}$$

k_n is the propagation constant of the lowest layer, and

$$\bar{E}_n^+ = E_{nx}^+ \hat{e}_x + E_{ny}^+ \hat{e}_y = \frac{\omega}{k_n} \bar{B}_n^+ \times \hat{e}_z \tag{2.17}$$

and

$$\bar{B}_n^+ = B_{nx}^+ \hat{e}_x + B_{ny}^+ \hat{e}_y . \tag{2.18}$$

The present treatment makes all charges vanish in the earth including the boundary surfaces, as of course, also those in the air (see Sections B.5, B.6 and B.7). The behaviour of the true charge density inside a conductor (the earth) has been discussed a little more generally by PRICE, 1967, p. 265, and by NEVANLINNA and WELLING, 1975, pp. 6–7.

According to Section B.7 the tangential components of the electric field intensity $\bar{E}$ and of the magnetic field intensity $\bar{H}$ will be continuous at the planes $z = 0$, $z = h_1$, $z = h_1 + h_2, \ldots, z = h_1 + h_2 + \ldots + h_{n-1}$ (see equations (B.57) and (B.59)). Hence taking equation (B.22) into account:

$$E_{0x}^+ + E_{0x}^- = E_{1x}^+ + E_{1x}^-, \tag{2.19}$$

$$E_{0y}^+ + E_{0y}^- = E_{1y}^+ + E_{1y}^- \tag{2.20}$$

$$\frac{1}{\mu_0} B_{0x}^+ + \frac{1}{\mu_0} B_{0x}^- = \frac{1}{\mu_1} B_{1x}^+ + \frac{1}{\mu_1} B_{1x}^- , \tag{2.21}$$

$$\frac{1}{\mu_0} B_{0y}^+ + \frac{1}{\mu_0} B_{0y}^- = \frac{1}{\mu_1} B_{1y}^+ + \frac{1}{\mu_1} B_{1y}^- \tag{2.22}$$

for $z = 0$,

$$E_{jx}^+ e^{-ik_j z_j} + E_{jx}^- e^{ik_j z_j} = E_{j+1x}^+ e^{-ik_{j+1} z_j} + E_{j+1x}^- e^{ik_{j+1} z_j} , \tag{2.23}$$

$$E_{jy}^+ e^{-ik_j z_j} + E_{jy}^- e^{ik_j z_j} = E_{j+1y}^+ e^{-ik_{j+1} z_j} + E_{j+1y}^- e^{ik_{j+1} z_j} , \tag{2.24}$$

$$\frac{1}{\mu_j} B_{jx}^+ e^{-ik_j z_j} + \frac{1}{\mu_j} B_{jx}^- e^{ik_j z_j} = \frac{1}{\mu_{j+1}} B_{j+1x}^+ e^{-ik_{j+1} z_j} + \frac{1}{\mu_{j+1}} B_{j+1x}^- e^{ik_{j+1} z_j} , \tag{2.25}$$

$$\frac{1}{\mu_j} B_{jy}^+ e^{-ik_j z_j} + \frac{1}{\mu_j} B_{jy}^- e^{ik_j z_j} = \frac{1}{\mu_{j+1}} B_{j+1y}^+ e^{-ik_{j+1} z_j} + \frac{1}{\mu_{j+1}} B_{j+1y}^- e^{ik_{j+1} z_j} , \tag{2.26}$$

for $z = z_j = h_1 + h_2 + ... + h_j$ and $j = 1,..., n-2$, and

$$E^+_{n-1x} e^{-ik_{n-1}z_{n-1}} + E^-_{n-1x} e^{ik_{n-1}z_{n-1}} = E^+_{nx} e^{-ik_n z_{n-1}} , \tag{2.27}$$

$$E^+_{n-1y} e^{-ik_{n-1}z_{n-1}} + E^-_{n-1y} e^{ik_{n-1}z_{n-1}} = E^+_{ny} e^{-ik_n z_{n-1}} , \tag{2.28}$$

$$\frac{1}{\mu_{n-1}} B^+_{n-1x} e^{-ik_{n-1}z_{n-1}} + \frac{1}{\mu_{n-1}} B^-_{n-1x} e^{ik_{n-1}z_{n-1}} = \frac{1}{\mu_n} B^+_{nx} e^{-ik_n z_{n-1}} \tag{2.29}$$

$$\frac{1}{\mu_{n-1}} B^+_{n-1y} e^{-ik_{n-1}z_{n-1}} + \frac{1}{\mu_{n-1}} B^-_{n-1y} e^{ik_{n-1}z_{n-1}} = \frac{1}{\mu_n} B^+_{ny} e^{-ik_n z_{n-1}} \tag{2.30}$$

for $z = z_{n-1} = h_1 + h_2 + ... + h_{n-1}$. The continuity equations (2.19)–(2.30) constitute a system of 4n linear equations. As the E-constants E^+_{0x}, E^-_{0x}, E^+_{0y},..., E^+_{ny} can be expressed in terms of the B-constants B^+_{0x}, B^-_{0x}, B^+_{0y},..., B^+_{ny} according to equations (2.3), (2.7), (2.11), (2.13) and (2.17), the number of unknown quantities in equations (2.19)–(2.30) is 4n+2 (including the quantities B^+_{0x} and B^+_{0y} associated with the primary field). This means that two of the B-constants can be considered »known» and the others can be determined in terms of them.

Equations (2.19)–(2.30) are resolved into two identical sets of 2n equations, one containing only x-components and the other only y-components of the magnetic field $\bar{B}$. All x-components can then be expressed in the form

$$B^\pm_{jx} = A^\pm_j B^+_{0x} , \qquad (j = 0, 1,..., n) , \tag{2.31}$$

and similarly the y-components

$$B^\pm_{jy} = A^\pm_j B^+_{0y} , \qquad (j = 0, 1,..., n) . \tag{2.32}$$

The coefficients A^+_j and A^-_j, which are the same for both the x- and the y-components, depend on the number of layers in the earth, on their thicknesses and on the parameters σ_j, ϵ_j and μ_j. Note that $A^+_0 = 1$ and $A^-_n = 0$. Note also that it would, of course, be equally possible to eliminate the B-constants and only use the E-constants in the solution of equations (2.19)–(2.30).

One aim of the present work is to evaluate the potential differences at the earth's surface during a geomagnetic disturbance. Let us first express the electric field at the earth's surface as a function of the measured geomagnetic disturbance on the earth's surface. Now the total geomagnetic disturbance $\bar{B}_M$ on the earth's surface has, according to equations (2.2), (2.4), (2.6), (2.8), (2.31) and (2.32), the expression

$$\bar{B}_M = \bar{B}_M(t) = \bar{B}_i(z{=}0) + \bar{B}_r(z{=}0) = ((B^+_{0x} + B^-_{0x})\hat{e}_x + (B^+_{0y} + B^-_{0y})\hat{e}_y)e^{i\omega t}$$
$$= (1 + A^-_0)(B_{0x}\hat{e}_x + B_{0y}\hat{e}_y)e^{i\omega t} \ . \qquad (2.33)$$

This formula represents the magnetic field at the earth's surface on the upper side of it. Because the tangential component of the magnetic field $\bar{B}$ need not be continuous across boundary surfaces, the magnetic field at the earth's surface on the lower side of it may in principle differ from equation (2.33). However, the permeability of the earth may be assumed to be μ_0 (SCHMUCKER, 1970, p. 3; see also LAHIRI and PRICE, 1939, pp. 509–510, PRICE, 1962, p. 1908, WAIT, 1962, p. 516, ALBERTSON and VAN BAELEN, 1970, and NEVANLINNA and WELLING, 1975, p. 6), *i.e.* $\mu_1 = \mu_2 \dots = \mu_n = \mu_0$, and so it follows from the continuity of the tangential component of the magnetic field intensity $\bar{H}$ that the tangential component of $\bar{B}$ is also continuous across all surfaces $z = 0$, $z = z_1, \dots, z = z_{n-1}$ (*cf.* equations (2.21), (2.22), (2.25), (2.26), (2.29) and (2.30)).

The electric field at the earth's surface is obtained from equations (2.1), (2.3), (2.4), (2.5), (2.7), (2.8), (2.31), (2.32) and (2.33):

$$\bar{E}_M = \bar{E}_M(t) = \bar{E}_i(z{=}0) + \bar{E}_r(z{=}0) \qquad (2.34)$$
$$= (c(B^+_{0y} - B^-_{0y})\hat{e}_x - c(B^+_{0x} - B^-_{0x})\hat{e}_y)e^{i\omega t}$$
$$= c(1 - A^-_0)(B^+_{0y}\hat{e}_x - B^+_{0x}\hat{e}_y)e^{i\omega t} = c\,\frac{1 - A^-_0}{1 + A^-_0}\,(B_{My}\hat{e}_x - B_{Mx}\hat{e}_y) \ ,$$
$$= c\,\frac{1 - A^-_0}{1 + A^-_0}\,\bar{B}_M \times \hat{e}_z \ .$$

Since the tangential component of the electric field $\bar{E}$ is continuous, $\bar{E}_M$ is the same on the upper side and on the lower side of the earth's surface.

The relationship between the electric and magnetic fields on the lower side of the earth's surface can be obtained from equations (2.23)–(2.30), which do not depend on the electromagnetic properties of the air or on the primary field. Due to the boundary conditions this relationship, which is thus also independent of the properties of the air, yields equation (2.34), if the electric and the magnetic fields on the lower side of the earth's surface are expressed by $\bar{E}_M$ and $(\mu_1/\mu_0)\bar{B}_M$, respectively. Hence the relationship between the electric field $\bar{E}_M$ and the magnetic field $\bar{B}_M$ on the upper side of the earth's surface given by formula (2.34) has to be independent of the assumptions that the air is non-conducting and has free space permittivity, but any values of these parameters are in principle allowable; only the free space permeability is assumed. This will be dealt with again in a more general manner in Section 2.7.

In the model in question both $\bar{B}_M$ and $\bar{E}_M$ are independent of the location on the surface of the earth and both are horizontal.

Let $P_1 = (x_1, y_1, 0)$ and $P_2 = (x_2, y_2, 0)$ be two points on the earth's surface. According to the »convention» made in Chapter 1 the potential difference between them is obtained by integration along the shortest path on the earth's surface, *i.e.* along a straight line, so that

$$U_{P_1P_2}(t) = \int_{P_1 \text{ straight line}}^{P_2} \bar{E} \cdot d\bar{l} = \int_{P_1 \text{ s.l.}}^{P_2} \bar{E}_M \cdot d\bar{l} = \bar{E}_M \cdot \bar{l} \tag{2.35}$$

where $\bar{l} = (x_2 - x_1)\hat{e}_x + (y_2 - y_1)\hat{e}_y$. Equation (2.35) defines $U_{P_1P_2}$ as the potential drop when moving from P_1 to P_2. Using equation (2.34)

$$U_{P_1P_2}(t) = c\,\frac{1 - A_0^-}{1 + A_0^-}\,(B_{My}(t)(x_2 - x_1) - B_{Mx}(t)(y_2 - y_1)) \tag{2.36}$$

$$= B_{Ml}(t)\; cl\,\frac{A_0^- - 1}{A_0^- + 1}$$

where l is the length of $\bar{l}$ and

$$B_{Ml}(t) = \frac{B_{Mx}(t)(y_2 - y_1) - B_{My}(t)(x_2 - x_1)}{l} = \bar{B}_M(t) \cdot \frac{(y_2 - y_1)\hat{e}_x - (x_2 - x_1)\hat{e}_y}{l} \tag{2.37}$$

is the scalar component of $\bar{B}_M$ in the (horizontal) direction $[(y_2 - y_1)\hat{e}_x - (x_2 - x_1)\hat{e}_y]/l$ perpendicular to $\bar{l}$. The unit vectors $\hat{e}_z$, $[(y_2 - y_1)\hat{e}_x - (x_2 - x_1)\hat{e}_y]/l$ and $\bar{l}/l$ make a right-handed system.

Let us express the complex field $\bar{B}_M$ as

$$\bar{B}_M = B_{Mx}\hat{e}_x + B_{My}\hat{e}_y = B_1 e^{i(\omega t + \varphi_1)}\hat{e}_x + B_2 e^{i(\omega t + \varphi_2)}\hat{e}_y\,, \tag{2.38}$$

where $B_1 = |B_{Mx}|$, $\omega t + \varphi_1 = \arg B_{Mx}$, $B_2 = |B_{My}|$ and $\omega t + \varphi_2 = \arg B_{My}$. Referring to Section A.1 it is then seen from equations (2.34), (2.36) and (2.38) that the physical electric field and the physical geomagnetic disturbance on the earth's surface and the physical potential difference between P_1 and P_2 are

$$\bar{E}_{M\,phys} = \frac{c\eta_-}{\eta_+}\,(-B_2\cos(\omega t + \varphi_2 + \varphi_- - \varphi_+)\hat{e}_x + B_1\cos(\omega t + \varphi_1 + \varphi_- - \varphi_+)\hat{e}_y)\,, \tag{2.39}$$

$$\bar{B}_{M\,phys} = B_1 \cos(\omega t + \varphi_1)\hat{e}_x + B_2 \cos(\omega t + \varphi_2)\hat{e}_y \tag{2.40}$$

and

$$U_{P_1 P_2\,phys} = \frac{c\eta_-}{\eta_+}\left(B_1(y_2 - y_1)\cos(\omega t + \varphi_1 + \varphi_- - \varphi_+) - B_2(x_2 - x_1)\cos(\omega t + \varphi_2 + \varphi_- - \varphi_+)\right) \tag{2.41}$$

The quantity η_- is the modulus and the quantity φ_- the argument of $A_0^- - 1$. Similarly η_+ and φ_+ are the modulus and argument of $A_0^- + 1$. Hence, expressed as a formula,

$$A_0^- \pm 1 = \eta_\pm e^{i\varphi_\pm} . \tag{2.42}$$

Using equations (2.39)–(2.41) the electric field and the potential differences occurring at the earth's surface can be estimated as a function of the geomagnetic disturbance. However, in practice the determination of A_0^- from equations (2.19)–(2.30) is laborious to carry out by hand, unless n is small. Formulae (2.39) and (2.40) show that the physical fields $\bar{E}_{M\,phys}$ and $\bar{B}_{m\,phys}$ are generally not perpendicular, though equation (2.34) valid for complex fields might suggest perpendicularity.

2.3. Induction in a homogeneous earth

Let us now discuss the simplest case involved in Section 2.2, in which the earth is taken to be homogeneous (n=1). The electromagnetic field within the earth only contains a downward travelling wave (cf. equations (2.15) and (2.16)), and the only continuity equations are

$$E_{0x}^+ + E_{0x}^- = E_{1x}^+ , \tag{2.43}$$

$$E_{0y}^+ + E_{0y}^- = E_{1y}^+ , \tag{2.44}$$

$$\frac{1}{\mu_0} B_{0x}^+ + \frac{1}{\mu_0} B_{0x}^- = \frac{1}{\mu} B_{1x}^+ \tag{2.45}$$

and

$$\frac{1}{\mu_0} B_{0y}^+ + \frac{1}{\mu_0} B_{0y}^- = \frac{1}{\mu} B_{1y}^+ \tag{2.46}$$

for $z = 0$. The subscript 1 has been omitted from μ_1, and likewise σ, ϵ and k will be used for the earth. By expressing the E-constants in terms of the B's (formulas (2.3), (2.7) and (2.17)), we easily obtain:

$$B_{0x}^{-} = \frac{\mu_0 kc - \mu\omega}{\mu_0 kc + \mu\omega} B_{0x}^{+} \tag{2.47}$$

and

$$B_{0y}^{-} = \frac{\mu_0 kc - \mu\omega}{\mu_0 kc + \mu\omega} B_{0y}^{+} \tag{2.48}$$

Hence

$$A_0^{-} = \frac{\mu_0 kc - \mu\omega}{\mu_0 kc + \mu\omega} . \tag{2.49}$$

Substitution of formula (2.49) into equation (2.34) gives

$$\bar{E}_M = \frac{\mu\omega}{\mu_0 k} (B_{My}\hat{e}_x - B_{Mx}\hat{e}_y) \tag{2.50}$$

or

$$\frac{E_{My}}{B_{Mx}} = -\frac{E_{Mx}}{B_{My}} = -\frac{\mu\omega}{\mu_0 k} . \tag{2.51}$$

The validity of equations (2.50) and (2.51) can be seen directly from formula (2.17) and from the fact that the electric field is equal on both sides of the earth's surface while the corresponding magnetic field values differ by a coefficient μ/μ_0.

As mentioned above it is reasonable to put μ equal to μ_0, and for the frequencies important in connection with geomagnetic variations the inequality $\sigma \gg \omega\epsilon$ is well satisfied in practice (see SARAOJA, 1946, pp. 122–123, KELLER and FRISCHKNECHT, 1970, pp. 52 and 203, KAUFMAN and KELLER, 1981, pp. 1–4, and Table I of this paper). The latter condition means that k can be approximated by $(1-i)/\delta = (\sqrt{2}/\delta)e^{-i\pi/4}$, where δ is the skin depth defined by equation (B.45). Thus the following formula is valid:

$$\frac{E_{My}}{B_{Mx}} = -\frac{E_{Mx}}{B_{My}} = -\frac{\omega\delta}{\sqrt{2}} e^{i\pi/4} = -\sqrt{\frac{\omega}{\mu_0\sigma}} e^{i\pi/4} . \tag{2.52}$$

This formula is a basic result of magnetotellurics (*cf.* CAGNIARD, 1953, p. 616).

Both equations (2.51) and (2.52) show that the increase of the conductivity of the earth tends to decrease the electric field at the earth's surface, which approaches zero as the conductivity approaches infinity.

Since the electric field and the magnetic field are coupled by Faraday's law of induction (B.3), it might seem natural to assume that an electric field component would be proportional to the time derivative of the orthogonal magnetic field component. This proportionality is suggested by WAIT, 1962, p. 512, and TAKALA, 1979, p. 27. Both refer to TIKHONOV, 1950, who examines a two-layer earth model with an infinitely conducting lower layer. Wait only speaks about the proportionality of the amplitudes of the quantities in question and specifies the statement for low frequencies. Since according to formula (2.38) the time derivative of B_{Mx} is

$$\frac{dB_{Mx}}{dt} = i\omega B_1 e^{i(\omega t+\varphi_1)} \tag{2.53}$$

equation (2.52) can be written as

$$\begin{aligned} -E_{My} &= \sqrt{\frac{\omega}{\mu_0\sigma}}\, B_{Mx} e^{i\pi/4} = \sqrt{\frac{\omega}{\mu_0\sigma}}\, B_1 e^{i(\omega t+\varphi_1+\pi/4)} \\ &= \frac{1}{\sqrt{\omega\mu_0\sigma}} \frac{dB_{Mx}}{dt}\, e^{-i\pi/4} \end{aligned} \tag{2.54}$$

E_{Mx} and B_{My} could, of course, equally well be considered. Equation (2.54) establishes that there is a phase shift of $\pi/4$ between $-E_{My}$ and dB_{Mx}/dt and that the proportionality coefficient between the amplitudes of $-E_{My}$ and dB_{My}/dt depends on ω, *i.e.* on the rate of the time variation of the fields. For these reasons the suggestion that the electric field and the time derivative of the orthogonal magnetic field are proportional is questionable. The same conclusion is also obtained directly from the investigation of the rigorous formula, where the proportionality coefficient between $-E_{My}$ and dB_{Mx}/dt is $\mu/i\mu_0 k$, which depends on ω through k. It is also seen from this proportionality factor that the phase shift is not exactly $\pi/4$, but depends on ω. In fact Tikhonov's formula involves an additional coefficient in the expression for E_{My}/B_{Mx}. This coefficient, which approaches unity as the thickness of the upper layer approaches infinity, *i.e.* as the earth becomes homogeneous, is a hyperbolic tangent function whose argument depends on ω (see Section 2.6). Thus the statement that E_{My} and dB_{Mx}/dt would be proportional can actually be considered even less valid in Tikhonov's treatment than in the case of a homogeneous earth. The relationship between E_{My} and the time-derivative of B_{Mx} will be

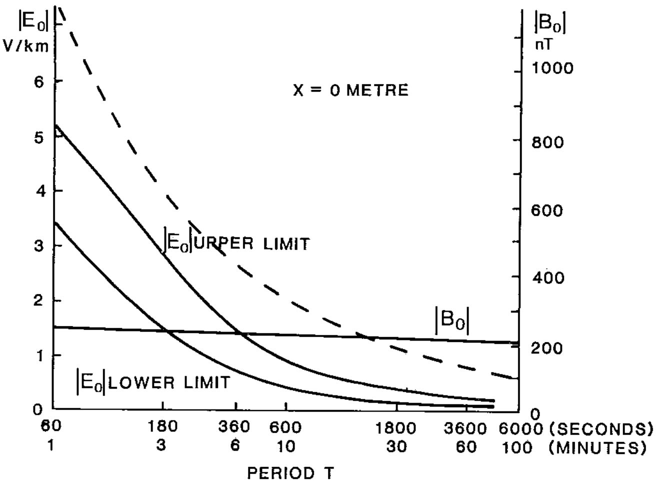

Fig. 2. Curve $|E_0|$, UPPER LIMIT represents the amplitude of the electric field on the earth's surface as a function of the time period T, when the primary field is a down plane wave, the structure of the earth is expressed by Table I and the amplitude of the magnetic field on the earth's surface is $|B_0|$. Curve $|E_0|$, LOWER LIMIT and the notation x = 0 METRE belong to a model in which the primary source is a line current. The dashed curve corresponds to $|E_0|$, UPPER LIMIT, but the earth is assumed to be homogeneous with a conductivity $10^{-4}\Omega^{-1}\mathrm{m}^{-1}$. The figure excluding the dashed curve is taken from ALBERTSON and VAN BAELEN, 1970.

discussed in Section 2.5 assuming a non-harmonic time-dependence.

The physical y-component of the electric field at the earth's surface has the following expression, which is obtained from formula (2.52) using equation (2.38):

$$E_{Myphys} = \frac{1}{\sqrt{2}}\,\delta\omega B_1 \cos\left(\omega t + \varphi_1 + \frac{5\pi}{4}\right). \tag{2.55}$$

The component E_{Mxphys} has a similar expression with B_1 and φ_1 replaced by B_2 and φ_2, respectively, and $5\pi/4$ by $\pi/4$.

The dashed curve in Fig. 2 depicts the amplitude of E_{Myphys} as a function of the time period $T = 2\pi/\omega$. The value $10^{-4}\Omega^{-1}\mathrm{m}^{-1}$ was chosen for σ, and B_1 was set equal to $|B_0|$ shown in the figure. The original figure belongs to ALBERTSON and VAN BAELEN, 1970, who have studied the electric field associated with geomagnetic disturbances using models in which the earth consists of eight horizontal layers, and is so outside the topic of this section. The thicknesses and the conductivities of the layers are shown in Table 1 (see Fig. 1). The permeability μ is everywhere equal to μ_0 and the permittivity ϵ has no effect. It is seen that the value of the

Table 1. The thicknesses and conductivities of the eight-layer earth model used by ALBERTSON and VAN BAELEN, 1970.

CONDUCTIVITY OF EARTH LAYERS		
Layer	Thickness (km)	Conductivity ($1/\Omega m$)
1	10	2×10^{-3}
2	390	1×10^{-4}
3	600	5×10^{-1}
4	500	1×10^{0}
5	500	1×10^{1}
6	500	1×10^{2}
7	400	5×10^{2}
8	∞	1×10^{4}

conductivity chosen for the dashed curve is the same as that of the second layer in Table 1.

Albertson's and Van Baelen's curve $|E_0|$, UPPER LIMIT represents the amplitude of the electric field at the earth's surface as a function of the period, when the primary field is a plane wave and the amplitude of the total magnetic field on the earth's surface is $|B_0|$. It can be seen that the dashed curve yields higher values than the curve $|E_0|$, UPPER LIMIT. This can evidently be interpreted as being due to the fact that the conductivity of the homogeneous earth is lower than that of the eight-layer model on the average. Such an explanation is, however, not always straightforward, since in some cases the so-called apparent conductivity, which describes the ratio of the perpendicular electric and magnetic field components on the earth's surface, may have a smaller value than the smallest conductivity of the layers or a higher value than the highest conductivity (CAGNIARD, 1953, p. 618, see also KAUFMAN and KELLER, 1981, pp. 75–104). For periods of less than 180s the relative difference between the dashed curve and the curve $|E_0|$, UPPER LIMIT is not very great (the ratio $\gtrsim 70$ %). This is due to the fact that the higher the frequency, the less significant the deep parts of the earth, where the two models differ considerably. For a conductivity of $10^{-4}\Omega^{-1}m^{-1}$ and a period of 180s, the skin depth is 675 km. If the period decreases below the values shown in Fig. 3, the curves obviously begin to differ more from each other again, since then the difference in the conductivity of the uppemost ten kilometers becomes more important.

The curve $|E_0|$, LOWER LIMIT, as well as the notation $x = 0$ METRE, belong to another model of ALBERTSON and VAN BAELEN, 1970, in which the primary source is a line current similar to that discussed in Chapter 3 of this paper.

Substitution of formula (2.49) into equation (2.36) gives the following expression for the potential difference between two points $P_1 = (x_1, y_1, 0)$ and $P_2 = (x_2, y_2, 0)$ on the surface of a homogeneous earth:

$$U_{P_1P_2}(t) = \frac{\mu\omega}{\mu_0 k}\,(B_{My}(t)(x_2 - x_1) - B_{Mx}(t)(y_2 - y_1))\,. \tag{2.56}$$

To simplify the expression let us assume that $\bar{l}$ points in the direction of the positive y-axis, *i.e.* $\bar{l} = (y_2 - y_1)\hat{e}_y$ with $y_2 > y_1$. (If this were not the case, the coordinate system could, of course, be suitably rotated.) Then $y_2 - y_1 = l$ and the potential difference is

$$U_{P_1P_2} = -\frac{\mu\omega l}{\mu_0 k} B_{Mx}\,. \tag{2.57}$$

This equation is equal to formula (2.51) (with E_{My} and B_{Mx}) multiplied by l. Similarly to equations (2.52) and (2.55) the following approximative formula can be considered valid:

$$U_{P_1P_2} = -\frac{1}{\sqrt{2}}\, l\delta\omega B_{Mx}\, e^{i\pi/4} \tag{2.58}$$

and

$$U_{P_1P_2\,phys} = \frac{1}{\sqrt{2}}\, l\delta\omega B_1 \cos\left(\omega t + \varphi_1 + \frac{5\pi}{4}\right). \tag{2.59}$$

Using the values $\omega = 1/60\text{s}$ (*i.e.* period $= 2\pi/\omega \approx 6$ min), $\omega B = 5\ nT/s$, $\sigma = 10^{-3}\Omega^{-1}\text{m}^{-1}$ and $l = 200$ km (*cf.* SARAOJA, 1946, p. 122, ALBERTSON and VAN BAELEN, 1970, and LANZEROTTI, 1979), equation (2.59) gives the amplitude of the voltage the value 220 V, which means about 1 V/km. This voltage gives rise to a current of 73 A in a power transmission line whose total impedance (for the frequency in question) is 3 Ω (*cf.* TAKALA, 1979, p. 43).

2.4. Skin depth rectangle

We shall in this section discuss the potential difference between points P_1 and P_2 on the earth's surface from another and more »practical» point of view, which I have not seen presented in the literature before. Define first a so-called skin depth rectangle so that its sides are the straight line l between P_1 and P_2 on the

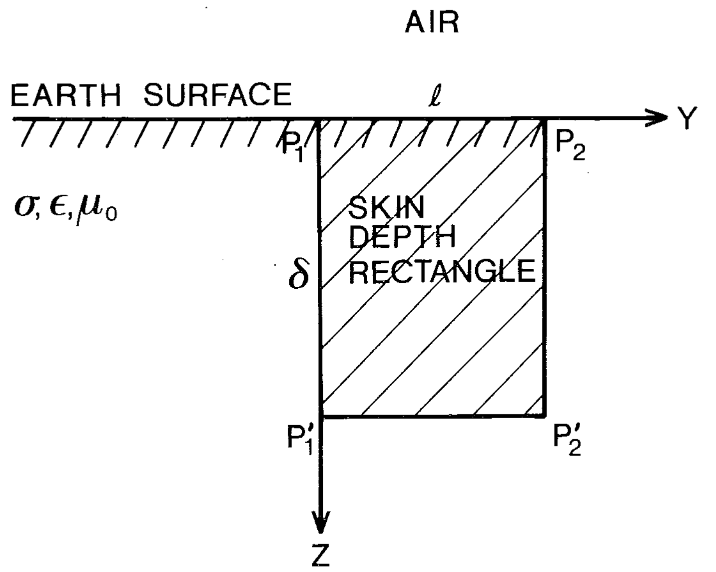

Fig. 3. Definition of skin depth rectangle.

earth's surface and the skin depth δ for the frequency discussed (Fig. 3). For simplification, let us again assume that the x-coordinates of P_1 and P_2 are the same.

The »apparent» magnetic flux through the skin depth rectangle can be defined as the product of the area of the rectangle and the perpendicular component of the magnetic field on the earth's surface. Thus, using formula (2.40) the apparent physical magnetic flux through the skin depth rectangle in the positive x-direction is

$$\phi_{a\,phys} = l\delta B_{Mx\,phys} = l\delta B_1 \cos(\omega t + \varphi_1) \tag{2.60}$$

According to Faraday's law of induction the negative time derivative of the magnetic flux through a circuit is equal to the (right-handed) induced electromotive force (or voltage) in the circuit (formula (B.54)). Therefore it is natural to define the »apparent» electromotive force $\epsilon_{a\,phys}(t)$ in the circuit $P_1 P_2 P_2' P_1'$ (in this direction, see Fig. 3) as

$$\epsilon_{a\,phys} = -\frac{d\phi_{a\,phys}}{dt} = l\delta\omega B_1 \sin(\omega t + \varphi_1) = l\delta\omega B_1 \cos\left(\omega t + \varphi_1 + \frac{3\pi}{2}\right) \tag{2.61}$$

It can be seen from equations (2.59) and (2.61) that the apparent electromotive force gives an estimate for the potential difference, because their amplitudes only differ by a factor of $\sqrt{2}$. In addition there is a phase shift of $\pi/4$ between $U_{P_1F_2phys}(t)$ and $\epsilon_{a\,phys}(t)$.

Notice that it is not necessary for this discussion of the apparent magnetic flux and of the apparent electromotive force for μ and μ_0 to be equal; equation (2.59) should simply be multiplied by μ/μ_0, in the definition of the apparent magnetic flux the magnetic field on the lower side of the earth's surface, *i.e.* $(\mu/\mu_0)\bar{B}_M$, must be used, and δ is the correct skin depth including μ. The definition of the apparent magnetic flux can also be extended to situations where the magnetic field is not constant in space on the earth's surface by using the average value along the line l.

2.5. Induction in a homogeneous earth with non-harmonic time-dependence

Let us now discuss induction in a homogeneous earth by neglecting the assumption of harmonic time-dependence. Assume, however, that the primary electromagnetic field is still transverse and only depends on z and t. Due to symmetry all other fields are also functions of z and t only, and evidently they are also transverse. Referring to the exact definition of a plane wave in Section B.6 all fields are thus plane waves. Assume further that the physical fields can be expressed as time-frequency-Fourier integrals according to equations (A.7) and (A.8). The y-component of the electric field and the x-component of the magnetic field on the earth's surface can be written as

$$E_{My}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_{My}(\omega) e^{i\omega t} d\omega = Re\left[\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} E_{My}(\omega) e^{i\omega t} d\omega\right] \qquad (2.62)$$

and

$$B_{Mx}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B_{Mx}(\omega) e^{i\omega t} d\omega = Re\left[\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} B_{Mx}(\omega) e^{i\omega t} d\omega\right] \qquad (2.63)$$

where

$$E_{My}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_{My}(t) e^{-i\omega t} dt \qquad (2.64)$$

and

$$B_{Mx}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B_{Mx}(t) e^{-i\omega t} dt\,. \qquad (2.65)$$

It follows from the reality of $E_{My}(t)$ and $B_{Mx}(t)$ that equation (A.14) must be satisfied by $E_{My}(\omega)$ and by $B_{Mx}(\omega)$. The latter equalities of formulas (2.62) and (2.63) are a consequence of equation (A.14), as shown in Section A.4 (equation (A.15)).

Since a direct application of the definition of the Fourier transform shows that Maxwell's equations are valid for each Fourier component of the fields and sources separately, assuming »sufficient» regularity of the fields and of the sources, it seems natural that the discussion about harmonic plane waves in Section 2.3 is valid for the behaviour of the Fourier components of the present electromagnetic field (cf. Chapter 5). Therefore it follows from equation (2.52) that under the assumptions $\mu = \mu_0$ and $\sigma \gg \omega\epsilon$

$$E_{My}(\omega) = -\sqrt{\frac{\omega}{\mu_0 \sigma}}\, e^{i\pi/4} B_{Mx}(\omega)\,. \tag{2.66}$$

Equation (2.52) was derived for $\omega > 0$ and so equation (2.66) is directly valid only for these values of ω. If the angular frequency is zero, *i.e.* there is no time dependence, it can be concluded that the electric field associated with the magnetic field is zero. Therefore the case $\omega = 0$ can formally be included in equation (2.66). It should also be borne in mind that the function $E_{My}(t)$ does not change even if the values of $E_{My}(\omega)$ are changed at separate points ω (Section A.4). This means that the value of $E_{My}(\omega{=}0)$ need not be specified exactly. As will soon be confirmed, negative values of ω can also be formally included in equation (2.66).

As already mentioned, equation (2.52) requires that the angular frequency ω is much smaller than the quantity σ/ϵ, the inverse of the relaxation time. So equation (2.66) cannot be used for all frequencies appearing in formulae (2.62) and (2.63), where the integration over ω goes to infinity. However, if the conductivity of the earth is of the order of $10^{-4} \ldots 10^{-3}\,\Omega^{-1}\mathrm{m}^{-1}$ and $\epsilon = 10\epsilon_0$, the quantity σ/ϵ has a value of the order of $10^{6} \ldots 10^{7}\ \mathrm{s}^{-1}$ (see SARAOJA, 1946, pp. 122–123). Then the frequencies for which equation (2.52) is not valid are really very high from a geomagnetic point of view (KELLER and FRISCHKNECHT, 1970, p. 203). Therefore the contribution of the values of ω for which equation (2.66) is not true is evidently negligible in the integrals

$$\int_0^\infty E_{My}(\omega) e^{i\omega t} d\omega \quad \text{and} \quad \int_0^\infty \sqrt{\frac{\omega}{\mu_0 \sigma}}\, e^{i\pi/4} B_{Mx}(\omega) e^{i\omega t} d\omega\,.$$

In other words, if ω_a is an upper limit of the positive angular frequencies for which equation (2.66) can be used, $E_{My}(t)$ can obviously be approximated as follows:

$$E_{My}(t) = Re\left[\sqrt{\frac{2}{\pi}}\int_0^\infty E_{My}(\omega)e^{i\omega t}d\omega\right] \tag{2.67}$$

$$\approx Re\left[\sqrt{\frac{2}{\pi}}\int_0^{\omega_a} E_{My}(\omega)e^{i\omega t}d\omega\right]$$

$$\approx Re\left[-\sqrt{\frac{2}{\pi}}\int_0^{\omega_a} \sqrt{\frac{\omega}{\mu_0\sigma}}\, e^{i\pi/4}B_{Mx}(\omega)e^{i\omega t}d\omega\right]$$

$$\approx Re\left[-\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{\omega}{\mu_0\sigma}}\, e^{i\pi/4}B_{Mx}(\omega)e^{i\omega t}d\omega\right] .$$

The validity of these approximations depends on the time t.

To be more accurate, however, let us use the exact formula (2.51), which gives equations (2.52) and (2.66) as an approximation. Assuming that μ is equal to μ_0 the rigorous formula is

$$E_{My}(\omega) = -\frac{\omega}{k(\omega)}B_{Mx}(\omega) . \tag{2.68}$$

For the time being this equation is only valid for positive angular frequencies. In the same way as equation (2.66) the value $\omega = 0$, for which $E_{My}(\omega)$ is equal to zero, can also be included in equation (2.68) (if the conductivity of the earth is not zero, which is an assumption in all discussions in this work). As mentioned, $E_{My}(\omega)$ and $B_{Mx}(\omega)$ satisfy equation (A.14). Therefore equation (2.68) is formally valid for negative values of ω, too, if only the argument of k is chosen to lie in the third quadrant of the complex plane, *i.e.* $-\pi \leqslant \arg k(\omega) = \arg\sqrt{\omega^2\mu_0\epsilon - i\omega\mu_0\sigma} \leqslant -3\pi/4$ (cf. inequalities (B.42)). This can also be seen by making the treatments above with an assumption of a time-dependence $e^{-i\omega t}$ ($\omega > 0$). The validity of equation (2.66) for negative angular frequencies is satisfied, requiring analogously that $\sqrt{\omega}$ is on the negative imaginary axis, *i.e.* $\arg\sqrt{\omega} = -\pi/2$.

Let us denote $dB_{Mx}(t)/dt$ by $g(t)$. Then the Fourier transform of $g(t)$ is

$$g(\omega) = i\omega B_{Mx}(\omega) \tag{2.69}$$

(see equation (A.9)). Hence equation (2.68) can be written as

$$E_{My} = \frac{ig(\omega)}{k(\omega)} . \tag{2.70}$$

Equations (2.62) and (2.70) give that

$$E_{My}(t) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{g(\omega)}{k(\omega)} e^{i\omega t} d\omega = \sqrt{\frac{2}{\pi}} Re\left[\int_0^{\infty} \frac{ig(\omega)}{k(\omega)} e^{i\omega t} d\omega\right]. \tag{2.71}$$

The expression of $E_{My}(t)$ will now be written in a new form using the convolution theorem (formula (A.13)). With the employment of equation (A.70)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{ie^{i\omega t}}{k(\omega)} d\omega = -\sqrt{\frac{2\pi}{\mu_0 \epsilon}}\, e^{-\sigma t/2\epsilon} J_0\left(i\,\frac{\sigma t}{2\epsilon}\right)\theta(t)\,. \tag{2.72}$$

In this formula, J_0 denotes the Bessel function of the zeroth order (see Section A.6), $\theta(t)$ is the step function defined in Section A.3 and $\sigma \neq 0$. Equation (2.72) indicates that $i/k(\omega)$ is the Fourier transform of the function $c(t) = -\sqrt{2\pi/\mu_0\epsilon}\,\cdot$ $e^{-\sigma t/2\epsilon} J_0(i\sigma t/2\epsilon)\theta(t)$, and inversely it is possible to show that formula (A.7) is also satisfied. From equations (2.71) and (A.13) it follows that

$$E_{My}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t-u)c(u)du \tag{2.73}$$

$$= -\frac{1}{\sqrt{\mu_0 \epsilon}} \int_0^{\infty} g(t-u) e^{-\sigma u/2\epsilon} J_0\left(i\,\frac{\sigma u}{2\epsilon}\right) du\,.$$

Since $g(t)$ is the time derivative of $B_{Mx}(t)$, equation (2.73) shows the dependence between $E_{My}(t)$ and $dB_{Mx}(t)/dt$, which is not a simple proportionality. We see that $E_{My}(t)$ is only influenced by the values of $g(t)$ before and at time t, which is natural because of causality, or in other words, the fact that the function $c(u)$ is zero for $u < 0$ agrees with causality. If u is very large compared to ϵ/σ, the function $J_0(i\sigma u/2\epsilon)$ can be replaced by the first term of the asymptotic expansion given by formulae (A.50) and (A.54)–(A.56):

$$J_0\left(i\,\frac{\sigma u}{2\epsilon}\right) \approx \left(\frac{\pi i \sigma u}{4\epsilon}\right)^{-1/2} \cos\left(i\,\frac{\sigma u}{2\epsilon} - \frac{\pi}{4}\right) \approx \sqrt{\frac{\epsilon}{\pi \sigma u}}\, e^{\sigma u/2\epsilon}\,. \tag{2.74}$$

The larger σ/ϵ, the smaller the values of u that can thus be included in this approximative formula. Let us substitute the right-hand side of formula (2.74) for $J_0(i\sigma u/2\epsilon)$ in the integral of equation (2.73) in spite of the fact that the integration starts from $u = 0$. Then

$$E_{My}(t) = -\frac{1}{\sqrt{\pi \mu_0 \sigma}} \int_0^{\infty} \frac{g(t-u)}{\sqrt{u}} du\,, \tag{2.75}$$

which is exactly the result that the use of equation (2.66) instead of (2.68) would have given (*cf.* CAGNIARD, 1953, p. 611). This is natural since both equation (2.66) for all values of ω and equation (2.75) are rigorously valid if σ/ϵ is infinitely large. If the error made in the neighbourhood of the point $u = 0$ in the integral can be proved negligible, equation (2.75) can be used.

Now we consider an example in which the magnetic field is assumed to be

$$B_{Mx}(t) = B_0\theta(t)e^{-\eta t} \tag{2.76}$$

where B_0 and η are real and positive constants and $\theta(t)$ is, as above, the step function. Equation (2.76) implies that there is an abrupt change in B_{Mx} at time $t = 0$ and as time elapses the magnetic field returns to its initial value (= 0). The change at time $t = 0$, which is discontinuous, is only a rough idealization for a relatively rapid but continuous physical change in the magnetic field. This discontinuity can also be regarded as questionable from the viewpoint of the assumption mentioned in section that the quantities vary continuously with time if Maxwell's equations are used. One solution to this difficulty is to assume that $B_{Mx}(t)$ is only arbitrarily close to formula (2.76) but continuous with also a continuous derivative. In practice the highest values of the time derivative of a geomagnetic field component are of the order of 10 nT/s (see LANZEROTTI, 1979).

It follows from equation (2.76) and Section A.3 that

$$g(t) = \frac{dB_{Mx}}{dt} = B_0\delta(t)e^{-\eta t} - B_0\eta\theta(t)e^{-\eta t}\ , \tag{2.77}$$

where $\delta(t)$ is the Dirac delta function. Substitution of equation (2.77) in equation (2.73) and the use of equations (A.4) and (A.5) give

$$E_{My}(t) = \begin{cases} -\dfrac{B_0}{\sqrt{\mu_0\epsilon}}\, e^{-\sigma t/2\epsilon} J_0\left(i\,\dfrac{\sigma t}{2\epsilon}\right) + \dfrac{B_0\eta e^{-\eta t}}{\sqrt{\mu_0\epsilon}} \displaystyle\int_0^t e^{-(\sigma/2\epsilon-\eta)u} J_0\left(i\,\dfrac{\sigma u}{2\epsilon}\right) du, & t>0 \\ -\dfrac{B_0}{2\sqrt{\mu_0\epsilon}}\ , & t=0 \\ 0\ , & t<0 \end{cases} \tag{2.78}$$

If equation (2.75) is used, the result is

$$E_{My}(t) = \begin{cases} -\dfrac{B_0}{\sqrt{\pi\mu_0\sigma t}} + \dfrac{B_0\eta e^{-\eta t}}{\sqrt{\pi\mu_0\sigma}} \displaystyle\int_0^t \dfrac{e^{\eta u}}{\sqrt{u}}\, du\,, & t > 0 \\ -\infty\,, & t = 0 \\ 0\,, & t < 0\,. \end{cases} \tag{2.79}$$

In the limit where η approaches zero, equations (2.78) and (2.79) can simply be written as

$$E_{My}(t) = -\frac{B_0}{\sqrt{\mu_0\epsilon}}\, e^{-\sigma t/2\epsilon} J_0\left(i\,\frac{\sigma t}{2\epsilon}\right)\theta(t) \tag{2.80}$$

and

$$E_{My}(t) = -\frac{B_0}{\sqrt{\pi\mu_0\sigma t}}\,\theta(t) \tag{2.81}$$

respectively. Equation (2.81) involves the same result as WAIT's, 1954, equation (8) with $\beta = 1$. The subscript 1 of σ_1 and also the quantity ΔH_0 are missing from Wait's expression for $e_x(t)$ and the summation sign is in the wrong place in his equation (8). Using formula (2.74) it can be seen that equation (2.80) approximately reduces to equation (2.81) if the condition $\sigma t/2\epsilon \gg 1$ is satisfied. The difference between equations (2.80) and (2.81) is greater, the earlier but non-negative, the time t.

2.6. Induction in a two-layered earth

In this section the special case n = 2 of Section 2.2. is discussed. Formulae (2.19)–(2.22) and (2.27)–(2.30), whose total number is eight, now constitute all boundary conditions. The quantity A_0^- can be obtained from these equations and the result is

$$A_0^- = \frac{\mu_0 k_1 c\alpha_1 - \mu_1 \omega\alpha_2}{\mu_0 k_1 c\alpha_1 + \mu_1 \omega\alpha_2} \tag{2.82}$$

where

$$\alpha_1 = 1 + \frac{\mu_1 k_2 - \mu_2 k_1}{\mu_1 k_2 + \mu_2 k_1}\, e^{-2ik_1 h} \tag{2.83}$$

and

$$\alpha_2 = 1 - \frac{\mu_1 k_2 - \mu_2 k_1}{\mu_1 k_2 + \mu_2 k_1} e^{-2ik_1 h} \tag{2.84}$$

omitting the subscript 1 from h. Substitution of formula (2.82) into equation (2.34) yields

$$\bar{E}_M = \frac{\mu_1 \omega \alpha_2}{\mu_0 k_1 \alpha_1} (B_{My} \hat{e}_x - B_{Mx} \hat{e}_y) \tag{2.85}$$

or

$$\frac{E_{My}}{B_{Mx}} = -\frac{E_{Mx}}{B_{My}} = -\frac{\mu_1 \omega \alpha_2}{\mu_0 k_1 \alpha_1} . \tag{2.86}$$

Equations (2.83), (2.84) and (2.86) involve the same result as WAIT's, 1954, formulae (2) and (3).

If both layers are assumed to have equal electromagnetic parameters making the earth homogeneous, α_1 and α_2 become equal to unity, and so formulae (2.85) and (2.86) reduce, as expected, to equations (2.50) and (2.51), respectively. Another means to achieve the case of a homogeneous earth is to let h grow infinitely large. This also makes α_1 and α_2 equal to unity, assuming that the conductivity σ_1 of the upper layer of the earth is not zero. If $\sigma_1 = 0$, the limit $h \to \infty$ does not reduce the situation to that treated in Section 2.3., since the influence of the lower layer is always seen because the field is not attenuated in the upper layer. So the upward propagating field does not vanish in the upper layer even in the limit. The formulae for a homogeneous earth are also obtained with the substitution $h = 0$.

If the conductivity σ_2 of the lower layer approaches infinity, α_1 and α_2 approach $1 + e^{-2ik_1 h}$ and $1 - e^{-2ik_1 h}$, respectively, and equation (2.86) becomes

$$\frac{E_{My}}{B_{Mx}} = -\frac{E_{Mx}}{B_{My}} = -\frac{\mu_1 \omega}{\mu_0 k_1} \tanh(ik_1 h) . \tag{2.87}$$

This is exactly the situation discussed by TIKHONOV, 1950, and mentioned in Section 2.3 (see also WAIT, 1962, p. 512). If h now approaches zero, $\bar{E}_M$ also goes to zero, which is natural, since the horizontal electric field has to be zero at the surface of the infinitely conducting layer.

Equations (2.83) and (2.84) show that if h is much larger than the inverse of

$|\mathrm{Imk}_1|$, the approximations $\alpha_1 \approx \alpha_2 \approx 1$ can be made. Formula (2.86) then indicates that the lower layer has no influence. In other words, the electromagnetic field is so efficiently damped in the upper layer that the lower layer is not sensed.

Let us assume that μ_1 and μ_2 are equal to μ_0 and that both inequalities $\sigma_1 \gg \omega\epsilon_1$ and $\sigma_2 \gg \omega\epsilon_2$ are satisfied. Formulae (2.83), (2.84) and (2.86) can then be expressed as

$$\alpha_1 = 1 + \frac{\delta_1 - \delta_2}{\delta_1 + \delta_2} e^{-2h/\delta_1} e^{-2ih/\delta_1} , \tag{2.88}$$

$$\alpha_2 = 1 - \frac{\delta_1 - \delta_2}{\delta_1 + \delta_2} e^{-2h/\delta_1} e^{-2ih/\delta_1} \tag{2.89}$$

and

$$\frac{E_{My}}{B_{Mx}} = -\frac{E_{Mx}}{B_{My}} = -\frac{\omega\delta_1}{\sqrt{2}} \frac{\alpha_2}{\alpha_1} e^{i\pi/4} = -\sqrt{\frac{\omega}{\mu_0\sigma_1}} \frac{\alpha_2}{\alpha_1} e^{i\pi/4} , \tag{2.90}$$

where the skin depths of the layers of the earth are denoted by δ_1 and δ_2 (*cf.* equation (2.52)). The result of equations (2.88)–(2.90) is also given by CAGNIARD's, 1953, formulae (28)–(30). Considering the equation $k_1 = (1-i)/\delta_1$, which was assumed valid in the derivation of formulae (2.88)–(2.90), the assumption $h \gg |\mathrm{Imk}_1|^{-1}$ treated above simply means that h is much larger than the skin depth of the upper layer.

Formulae (2.88)–(2.90) and also (2.83), (2.84) and (2.86) are consistent with the fact that the electric field $\bar{E}_M$ approaches zero as σ_1 approaches infinity, though the exact dependence of $\bar{E}_M$ on σ_1 is not as straightforward as in the case of a homogeneous earth. The thickness h of the upper layer does not affect the result $\lim_{\sigma_1\to\infty} \bar{E}_M = 0$ (provided $h \neq 0$), and so h may be decreased arbitrarily without changing the first limit, *i.e.*

$$\lim_{h\to 0} \left(\lim_{\sigma_1\to\infty} \bar{E}_M \right) = 0 . \tag{2.91}$$

If the same limit processes are made in the reverse order the result is not the same since, as indicated above, the limit $h \to 0$ (with $\sigma_1 \neq \infty$) makes the influence of the upper layer vanish, and

$$\lim_{\sigma_1\to\infty} \left(\lim_{h\to 0} \bar{E}_M \right) = \frac{\mu_2\omega}{\mu_0 k_2} (B_{My}\hat{e}_x - B_{Mx}\hat{e}_y) . \tag{2.92}$$

The observation obtained here that the »double-limit» depends on the order of the limit processes is not exceptional; another similar case is shown in Section 3.3. Two more examples of the same phenomenon are given by D'ERCEVILLE and KUNETZ, 1962, p. 657, and PIRJOLA, 1975, pp. 53–54. However, it seems very difficult to perform one of d'Erceville's and Kunetz's limits ($\omega \to 0$) in practice using the formula given by them on page 665.

2.7. Induction in an earth having arbitrarily changing properties in the vertical direction

Formulae (2.51), (2.52), (2.86) and (2.90) imply basic principles of geomagnetic induction in the earth, and the treatment of models where the earth consists of more than two layers is neglected in this work referring to the general equations represented in Section 2.2. Three-layer earth models are discussed by CAGNIARD, 1953, and WAIT, 1962. The eight-layer model treated by ALBERTSON and VAN BAELEN, 1970, was mentioned in Section 2.3.

The induction problem discussed in this chapter can be formulated in a still more general form by assuming that the parameters σ, ϵ and μ of the earth are arbitrary functions of depth. The conductivity is, however, assumed to be finite everywhere. The tangential component of $\bar{H}$ is then continuous (see formulas (B.58) and (B.59)). It should be noted, however, that the treatment would also be equally possible, and actually easier, with infinite conductivities. Lateral variations in the parameters would completely change the treatment, because the assumption valid now that the only space-dependence of the fields is with respect to z would not be true, and lateral variations are not accepted here. As above, the air is regarded as electromagnetically free space, the primary electromagnetic field is expressed by formulae (2.1)–(2.4) and the secondary field in the air by equations (2.5)–(2.8). Since the time-dependence is harmonic, it would seem tempting to think that the field in the earth, excluding the possible planes of discontinuity, is simply obtained from wave equations (B.43) and (B.44), provided that the z-dependence of k is taken into account. This is not, however, correct, because the derivation of the wave equations assumes the parameters to be constant. Therefore, the original Maxwell equations must be considered.

Maxwell's equations (B.12) and (B.13) yield

$$\nabla \times \bar{E} = -i\omega\mu\bar{H} \tag{2.93}$$

and

$$\nabla \times \bar{H} = (\sigma + i\omega\epsilon)\bar{E} \tag{2.94}$$

where formulae (B.21), (B.22) and (B.25) have been used. Equation (2.94) cannot be written in the form of formula (B.24), because μ may have space dependence. Taking the divergence of equations (2.93) and (2.94) gives

$$\nabla \cdot \bar{H} = -\frac{1}{\mu}\nabla\mu \cdot \bar{H} \tag{2.95}$$

and

$$\nabla \cdot \bar{E} = -\frac{1}{\sigma + i\omega\epsilon}\nabla(\sigma + i\omega\epsilon) \cdot \bar{E} \tag{2.96}$$

Equations (2.93)–(2.96) are Maxwell's equations for the present problem. Formula (2.95) is completely equivalent with equation (B.2), and formula (2.96) compared with equation (B.1) shows that the total charge density is $-\epsilon_0/(\sigma+i\omega\epsilon)\cdot \nabla(\sigma+i\omega\epsilon)\cdot\bar{E}$. Formulae (B.10), (B.21) and (2.96) give the expressions

$$\rho_{true} = \epsilon\nabla \cdot \bar{E} + \nabla\epsilon \cdot \bar{E} = \frac{\sigma\nabla\epsilon - \epsilon\nabla\sigma}{\sigma + i\omega\epsilon} \cdot \bar{E} = \frac{\sigma^2}{\sigma + i\omega\epsilon}\nabla\left(\frac{\epsilon}{\sigma}\right) \cdot \bar{E} \tag{2.97}$$

for the true charge density (see NEVANLINNA and WELLING, 1975, p. 7).

The curl of equation (2.93) and the use of equations (2.93), (2.94) and (2.96) yield the wave equation

$$\nabla^2\bar{E} + \nabla\left(\frac{\nabla(\sigma + i\omega\epsilon) \cdot \bar{E}}{\sigma + i\omega\epsilon}\right) + \frac{\nabla\mu \times (\nabla \times \bar{E})}{\mu} + k^2\bar{E} = 0\ . \tag{2.98}$$

The other wave equation is analogously obtained from equations (2.93)–(2.95):

$$\nabla^2\bar{H} + \nabla\left(\frac{\nabla\mu \cdot \bar{H}}{\mu}\right) + \frac{\nabla(\sigma + i\omega\epsilon) \times (\nabla \times \bar{H})}{\sigma + i\omega\epsilon} + k^2\bar{H} = 0\ . \tag{2.99}$$

Maxwell's equations (2.93)–(2.96) and wave equations (2.98) and (2.99) are valid at every point where the properties of the earth vary continuously. At points of discontinuity boundary conditions have to be used.

Formulae (2.93)–(2.99) are valid for any space variation of the parameters because the assumption of vertical variation only has not yet been utilized. Equations (2.98) and (2.99) show that adding the space-dependence to k is really not sufficient to make the »normal» wave equations (B.43) and (B.44) correct for the inhomogeneous case. ALBERTSON and VAN BAELEN, 1970, however, use the »normal» wave equation for the electric field inside the earth neglecting the displacement current and with the assumed z-dependence of the conductivity taken into account. This is correct, because Albertson and Van Baelen also assume that the charge density is zero in the earth, thus making the second term in equation (2.98) vanish; the third term is zero owing to their assumption that μ is equal to μ_0, *i.e.* constant.

Let us now make use of the fact, valid in this section, that only *z*-space-dependence is present. Equations (2.93) and (2.94) then show that the z-components of $\bar{H}$ and $\bar{E}$ are zero. This implies that the right-hand sides of both formula (2.95) and formula (2.96) vanish. Hence the total volume charge density in the earth is zero, and from equation (2.97) we see that this is true for all kinds of volume charge. No charge can appear on the planes of discontinuity *z*=constant, including the earth's surface, because $\bar{E}$ and $\bar{D}$ have zero z-components (see Section B.7). The wave equation (2.99) reduces to

$$\frac{\partial^2 \bar{H}}{\partial z^2} - \frac{1}{\sigma + i\omega\epsilon} \frac{d(\sigma + i\omega\epsilon)}{dz} \frac{\partial \bar{H}}{\partial z} + k^2 \bar{H} = 0 \, , \tag{2.100}$$

from which $\bar{H}$ $(=\bar{B}/\mu) = \bar{H}(z) \cdot e^{i\omega t}$ can be solved. Equation (2.100) is a linear and homogeneous differential equation of the second order with respect to z. Its general solution is

$$\bar{H} = \bar{H}^{+} f_{+}(z) e^{i\omega t} + \bar{H}^{-} f_{-}(z) e^{i\omega t} \tag{2.101}$$

where $\bar{H}^{+}$ and $\bar{H}^{-}$ are two constant complex vectors and $f_{+}(z)$ and $f_{+}(z)$ two linearly independent solutions of the differential equation. The notations + and – are used for analogy with Sections 2.1 and 2.2. The exact dependence of $\sigma + i\omega\epsilon$ and μ on z should be known to be able to study the functions $f_{+}(z)$ and $f_{-}(z)$ thoroughly. So the discussion of the directions of attenuation and of phase and energy propagations is omitted here (cf. Sections B.6 and B.9). As pointed out above, the *z*-component of $\bar{H}$ vanishes. Therefore the *z*-components of $\bar{H}^{+}$ and $\bar{H}^{-}$ will be equal to zero. We obtain from equations (2.94) and (2.101) that

$$\bar{E} = \frac{1}{\sigma + i\omega\epsilon} \hat{e}_z \times \bar{H}^{+} \frac{df_{+}(z)}{dz} e^{i\omega t} + \frac{1}{\sigma + i\omega\epsilon} \hat{e}_z \times \bar{H}^{-} \frac{df_{-}(z)}{dz} e^{i\omega t} . \tag{2.102}$$

The solution (equations (2.101) and (2.102)) was obtained utilizing the wave equation. Hence it is best to ascertain that Maxwell's equations (2.93)–(2.96) are really satisfied (*cf.* the comment after formulae (B.43) and (B.44)). The validity of equations (2.94) and (2.95) is already involved in the above discussion, and the former also shows that equation (2.96) is satisfied. The validity of formula (2.93) can be proved to follow from equations (2.94), (2.95) and (2.99).

If the earth has one or several planes of discontinuity $z=z_j$, $j=1,...,n-1$, *i.e.* n »continuous» horizontal regions, the solutions (2.101) and (2.102) must be obtained for each layer separately. The functions $f_+(z)$ and $f_-(z)$ are different in different regions. This could be acknowledged by denoting $f_{j-}(z)$ and $f_{j+}(z)$ ($j=1,...,n$). The solutions for different regions are joined together through boundary conditions. It seems reasonable that in the lowest continuous region only one fixed linear combination of the solutions $f_+(z)$ and $f_-(z)$ can be accepted for physical reasons. Let us assume that the accepted one is merely $f_+(z)$, for any fixed linear combination of the original functions $f_+(z)$ and $f_-(z)$ can be denoted by $f_+(z)$. So, as in Section 2.2, the total number of unknown coefficients in the earth is 4n–2. These are H^+_{1x}, H^+_{1y}, H^-_{1x}, H^-_{1y}, H^+_{2x},..., H^+_{nx} and H^+_{ny}. ($H^{\pm}_{jx/y}$ denotes the x/y-component of $\bar{H}^{\pm}$ in the j^{th} region.) The continuity conditions of $\bar{E}$ and $\bar{H}$ at the boundaries $z=z_j$ ($j=1,...,n-1$) give 4n–4 linear equations, which resolve into two identical sets of 2n–2 equations, one containing only H_x- and the other only H_y-quantities. The situation is thus very similar to the special case of homogeneous layers discussed in Section 2.2, and all coefficients in the earth can be expressed for example in terms of H^+_{1x} and H^+_{1y} as

$$H^{\pm}_{jx} = Y^{\pm}_j H^+_{1x} \qquad (j = 1,...,n) \tag{2.103}$$

and

$$H^{\pm}_{jy} = Y^{\pm}_j H^+_{1y} \qquad (j = 1,...,n)\,. \tag{2.104}$$

The coefficients Y^+_j and Y^-_j depend on the properties of the earth and are in principle known. The analogy between formulae (2.103) and (2.104) and equations (2.31) and (2.32) is not complete, since in this section only boundary conditions inside the earth have been treated.

The treatment of the magnetic flux density $\bar{B}$ instead of the magnetic field intensity $\bar{H}$ would certainly have been equally possible in the above discussion, but the latter was chosen because its treatment seemed simpler.

Using equations (2.101)–(2.104) and the fact that the tangential components of $\bar{E}$ and $\bar{H}$, *i.e.* in this case the total vectors $\bar{E}$ and $\bar{H}$, are continuous, the magnetic and electric fields at the earth's surface on its upper side have the following expressions:

$$\bar{B}_M = \bar{B}_M(t) = \mu_0(f_+ + Y^-_1 f_-)(H^+_{1x}\hat{e}_x + H^+_{1y}\hat{e}_y)\,e^{i\omega t} \tag{2.105}$$

and

$$\bar{E}_M = \bar{E}_M(t) = -\frac{1}{\sigma_1 + i\omega\epsilon_1}(f'_+ + Y_1^- f'_-)(H_{1y}^+ \hat{e}_x - H_{1x}^+ \hat{e}_y)e^{i\omega t} \tag{2.106}$$

$$= \frac{1}{\mu_0(\sigma_1 + i\omega\epsilon_1)} \frac{f'_+ + Y_1^- f'_-}{f_+ + Y_1^- f_-} \hat{e}_z \times \bar{B}_M \, .$$

In these equations σ_1 and ϵ_1 denote the values of the conductivity and the permittivity of the earth at the earth's surface, *i.e.* $\sigma_1 = \sigma(z = 0)$ and $\epsilon_1 = \epsilon(z = 0)$, f_+, f_-, f'_+ and f'_- are the values of the functions $f_+(z)$ and $f_-(z)$ connected with the uppermost layer and of their derivatives at $z = 0$, respectively. As in the case of a horizontally layered earth, $\bar{E}_M$ and $\bar{B}_M$, which have no vertical components, are constant all over the surface of the earth. The quantity $-\frac{1}{\sigma_1 + i\omega\epsilon_1}\frac{f'_+ + Y_1^- f'_-}{f_+ + Y_1^- f_-}$, which has the dimension of resistance (Ω) and is equal to the ratios $\mu_0 E_{Mx}/B_{My} = E_{Mx}/H_{My}$ and $-\mu_0 E_{My}/B_{Mx} = -E_{My}/H_{Mx}$, can be called the surface impedance at the earth's surface. It depends only on the properties of the earth. (In formulae (2.34), (2.50) and (2.85) the (implicit) surface impedances are $\mu_0 c \frac{1 - A_0^-}{1 + A_0^-}$, $\frac{\mu\omega}{k} \approx \sqrt{\frac{\mu\omega}{\sigma}}\, e^{i\pi/4}$ and $\frac{\mu_1 \omega \alpha_2}{k_1 \alpha_1} \approx \sqrt{\frac{\mu_1 \omega}{\sigma_1}}\, \frac{\alpha_2}{\alpha_1}\, e^{i\pi/4}$, respectively.)

The relationship between $\bar{E}_M$ and $\bar{B}_M$ is thus independent of the primary field and of the properties of the upper half-space, the air, whose electromagnetic parameters in principle need not even be constants, provided the permeability of the air at the earth's surface is denoted by μ_0 (see the comment after formula (2.34) in Section 2.2). So the assumptions made at the beginning of this section about the air and the primary field can be considered too limiting and unnecessary. However, a harmonic time-dependence and only a z-space-dependence of the field in the earth were assumed when deriving formula (2.106) (see Chapter 5).

If the original assumptions that the primary and secondary fields are given by equations (2.1)–(2.8) are valid, $\bar{E}_M$ and $\bar{B}_M$ must be equal to the vectors $c(\bar{B}_0^+ - \bar{B}_0^-) \times \hat{e}_z e^{i\omega t}$ and $(\bar{B}_0^+ + \bar{B}_0^-)e^{i\omega t}$, respectively. When these equalities are explicitly expressed and equations (2.4), (2.8), (2.105) and (2.106) are used, four linear equations with four unknowns B_{0x}^-, B_{0y}^-, H_{1x}^+ and H_{1y}^+ are obtained, if the vector $\bar{B}_0^+$ associated with the primary field is regarded as known. This set of equations is, analogously to the situations above, comprised of two identical pairs of equations, one containing only x- and the other only y-components. The former yields B_{0x}^- and H_{1x}^+ in terms of B_{0x}^+ and the latter B_{0y}^- and H_{1y}^+ in terms of B_{0y}^+. Then both $\bar{E}_M$ and $\bar{B}_M$ could be obtained as functions of the primary field and of other parameters.

The potential difference between two points on the earth's surface is given by equations (2.35) and (2.106).

2.8. Comment on Cagniard's remarks

CAGNIARD, 1953, gives an expression for the magnetic field on the earth's surface on page 608. This expression is based on Maxwell's equation (B.13) in the integral form neglecting the displacement current (Ampère's law) and on the assumptions of a harmonic time-dependence and of only a vertical space-dependence. Owing to the low frequencies compared with the relaxation time of the earth, the neglect of the displacement current is really permissible in connection with geomagnetic induction. In his remarks on page 609 Cagniard, on the other hand, expresses the magnetic field as an integral which occurs on the earth's surface and is caused by the horizontally-directed and laterally homogeneous currents induced within the earth. This integral is rigorous in the case of direct currents, but since, as mentioned, the frequencies are low, it is clearly applicable also in the treatment of geomagnetic induction.

The latter of the two magnetic fields expressed by Cagniard is half of the former. In his remarks Cagniard explains that the inequality of these fields is due to the use of the formula which is fully accurate only for direct currents. But the true reason for the difference is that the former field represents the total magnetic field, while the latter is the secondary, reflected magnetic field. These fields differ from each other by the magnitude of the primary magnetic field whatever the frequency of the time variations is. The fact that it is a question of two different fields also seems to be recognized by PRICE, 1962, p. 1911, and by KAUFMAN and KELLER, 1981, pp. 49–50, although they do not refer to Cagniard's remarks.

If the field due to the induced currents within the earth is calculated using Ampère's law, the contribution of the magnetic field at $z = \infty$ to the line integral does not vanish for any frequency as in Cagniard's calculation, where the total magnetic field is discussed.

3. *Induction in the case of a line current primary source oscillating harmonically in time*

3.1 Description of the model and the expressions of the primary field

As in Chapter 2, let us describe the earth as the lower half-space and the earth's surface as an infinite plane. The upper half-space, the air, is again assumed to behave electromagnetically as free space. At a later stage in this chapter some conductivity will be assigned to the air. As will be stated, this conductivity may be arbitrarily small. The primary electromagnetic field is now assumed to be caused by an infinitely long (true) straight line current situated parallel to and at some height above the earth's surface in the air and oscillating harmonically with time.

The Cartesian coordinate system in Chapter 2 is also used in this chapter, *i.e.* the x- and y-axes point northward and eastward, respectively, the z-axis points downward and the earth's surface is the plane $z = 0$. If the primary current flows in the direction of the y-axis in the plane $x = 0$, the expression for the current density is

$$\bar{j} = Je^{i\omega t}\delta(z + h)\delta(x)\hat{e}_y \tag{3.1}$$

where J is a complex constant implying the magnitude and the phase of the current, the δ's are delta functions (equations (A.3) and (A.4)), and $h(>0)$ is the height of the current from the earth's surface. The angular frequency ω is assumed to be positive (and the unit vector $\hat{e}_y$ points in the positive y-direction). The current density of equation (3.1) is »delta-type» – infinitely large at the line $x = 0$, $z = -h$ and discontinuous in the transverse direction, which are idealizations. Although the fixation of the coordinate system and equation (3.1) define the source current as parallel to the east-west direction, the treatment of this section is not confined to this case. If the direction of the current differs from the east-west direction, but is parallel to the earth's surface, the coordinate system has to be rotated.

The primary field, which is the electromagnetic field caused by the current of equation (3.1) in free space around it, can be calculated using formulae (B.80) and (B.81). The discussions in Sections B.1 and B.8 can be referred to concerning the point that the infiniteness and transverse discontinuity mentioned above are accepted in equations (B.80) and (B.81). It should also be noted that the current density given by equation (3.1) makes the integral $\int_S da'\hat{n}' \cdot [\bar{j}]/R$ vanish, as S approaches infinity. This, as stated in Section B.8, is necessary for the Lorenz condition (B.74) to be satisfied by formulae (B.78) and (B.79) and hence also for the use of equations (B.80) and (B.81).

Alternatively it is possible to derive expressions for the primary electromagnetic field assuming first a finite thickness to the current, solving Maxwell's equations inside and outside this current »tube», setting continuity conditions at the surface of the »tube» and letting the thickness of the »tube» finally go to zero. Let us, however, use a »modified version» of the former manner in which the primary electric field $\bar{E}$ is calculated from equation (B.80) and the primary magnetic field $\bar{B}$ from Maxwell's equation (B.3).

The divergence of the source current given by formula (3.1) vanishes, and so does not imply the existence of charge (equation (B.18)). In addition to the current of equation (3.1), there could be time-independent source charge, but then a new situation would be generated in which the origin of the whole phenomenon would no longer be the current of equation (3.1) alone. The existence of time-independent primary charge would also make the future assumption invalid, *i.e.* the only time-dependence occurring is $e^{i\omega t}$. (This last statement is true in analogous situations later in this work, too, when a time-independent or an exponentially damping primary charge is rejected.) So we assume that all primary charge is identically zero. Hence only the last integral in equation (B.80) remains and

$$\bar{E}(\bar{r},t) = -\frac{1}{4\pi\epsilon_0 c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\omega J e^{i\omega\left(t-\frac{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}{c}\right)}\delta(z'+h)\delta(x')\hat{e}_{y'}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}dx'dy'dz'$$

$$= -\frac{i\omega\mu_0 J e^{i\omega t}}{4\pi}\hat{e}_y\int_{-\infty}^{\infty}\frac{e^{-i\frac{\omega}{c}\sqrt{x^2+(z+h)^2+(y-y')^2}}}{\sqrt{x^2+(z+h)^2+(y-y')^2}}dy' \qquad (3.2)$$

where formula (A.4) has been employed. The unit vector $\hat{e}_y$ is the same for both the source point $\bar{r}' = (x', y', z')$ and the point of observation $\bar{r} = (x, y, z)$, *i.e.* $\hat{e}_y = \hat{e}_{y'}$. By denoting the real and positive quantity $\sqrt{x^2+(z+h)^2}$ by r_0 and changing the variable of integration to $u = y' - y$ the electric field can be expressed as

$$\bar{E}(\bar{r},t) = -\frac{i\omega\mu_0 J e^{i\omega t}}{2\pi}\hat{e}_y\int_0^{\infty}\frac{e^{-i\frac{\omega}{c}\sqrt{r_0^2+u^2}}}{\sqrt{r_0^2+u^2}}du \qquad (3.3)$$

If a new variable of integration α is introduced by $\alpha = \sqrt{1+u^2/r_0^2}$, equation (3.3) implies that

$$\bar{E}(\bar{r},t) = -\frac{i\omega\mu_0 J e^{i\omega t}}{2\pi}\hat{e}_y\int_1^{\infty}\frac{e^{-i\frac{\omega}{c}r_0\alpha}}{\sqrt{\alpha^2-1}}d\alpha \qquad (3.4)$$

Equations (3.3) and (3.4) show that $\bar{E}$ does not depend on y, which is natural for reasons of symmetry. The integral in equation (3.4) can be expressed in terms of the Hankel function of the second kind and of the zeroth order $H_0^{(2)}$ using equations (A.24), (A.29), (A.66) and (A.67); thus

$$\bar{E}(x, z, t) = -\frac{\omega\mu_0 J e^{i\omega t}}{4} H_0^{(2)}(k_0\sqrt{x^2 + (z+h)^2})\hat{e}_y \tag{3.5}$$

where formula (B.91) has been employed.

All fields can be assumed to have the time-dependence $e^{i\omega t}$ of the primary source expressed by formula (3.1). The justification of this assumption for the primary magnetic field is seen from formulae (B.81) and (3.1) without any integration. Thus, the primary magnetic field, using equations (B.3) and (3.5), is expressed by

$$\bar{B}(x, z, t) = -\frac{1}{i\omega} \nabla \times \bar{E}(x, z, t) \tag{3.6}$$

$$= \frac{i\mu_0 k_0 J e^{i\omega t}}{4\sqrt{x^2 + (z+h)^2}} H_1^{(2)}(k_0\sqrt{x^2 + (z+h)^2})(-(z+h)\hat{e}_x + x\hat{e}_z) .$$

The Hankel function of the second kind and of the first order is denoted by $H_1^{(2)}$, and formula (A.40) has been utilized.

For frequencies significant in connection with geomagnetic variations k_0 is small compared with the inverse of a reasonable value of $\sqrt{x^2 + (z+h)^2}$ for example for $\omega = 3s^{-1}$ k_0 is equal to 10^{-8}m^{-1} (see KAUFMAN and KELLER, 1981, pp. 2–3). Therefore formulae (A.48) and (A.49) for small arguments of the Hankel functions can be used in equations (3.5) and (3.6), which then have the forms

$$\bar{E}(x, z, t) = \frac{i\omega\mu_0 J e^{i\omega t}}{2\pi} \log(k_0\sqrt{x^2 + (z+h)^2})\hat{e}_y \tag{3.7}$$

and

$$\bar{B}(x, z, t) = \frac{\mu_0 J e^{i\omega t}}{2\pi\sqrt{x^2 + (z+h)^2}} \frac{(z+h)\hat{e}_x - x\hat{e}_z}{\sqrt{x^2 + (z+h)^2}} \tag{3.8}$$

The quantity $\sqrt{x^2 + (z+h)^2} = r_0$ gives the distance of the point of observation from the line current, and the vector $((z+h)\hat{e}_x - x\hat{e}_z)/\sqrt{x^2 + (z+h)^2}$ is the cylindrical unit vector $\hat{e}_\varphi$ around the line current. Hence the magnetic field is fairly accurately obtained from the formula which is valid for a time-independent straight line current and which is obtained from Maxwell's equation (B.4) with the term $\mu_0\epsilon_0\partial\bar{E}/\partial t$ equal to zero and using Stokes' theorem.

3.2 Induction in a horizontally layered earth

In a manner similar to that of Section 2.2, the earth is again assumed to consist of n horizontal layers (Fig. 1) with constant conductivities σ_j ($\neq\infty$), constant permittivities ϵ_j, constant permeabilities μ_j and thicknesses h_j ($h_n = \infty$). It was mentioned in Section 3.1 that the only time-dependence appearing is $e^{i\omega t}$, and due to symmetry all fields have to be independent of the y-coordinate.

The secondary field caused by the earth in the air satisfies Maxwell's equations (B.35)–(B.38) with $\sigma = 0$, $\epsilon = \epsilon_0$ and $\mu = \mu_0$. Thus no charge is associated with the secondary field in the air either. This is also true later when a small conductivity of the air is assumed. Formulae (B.43) and (B.44) with $k = k_0$ (equation (B.91)) are also valid. Then

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + k_0^2 E_y = 0 \,. \tag{3.9}$$

Equation (3.9) is solved in the form

$$E_y = f(x)g(z) \,, \tag{3.10}$$

where the time factor $e^{i\omega t}$ has not been written explicitly. This method of solving is known as separation of variables, see *e.g.* STRATTON, 1941, pp. 197–198, MORSE and FESHBACH, 1953, pp. 497–498. The substitution of equation (3.10) in equation (3.9) after manipulation makes

$$\frac{1}{f}\frac{d^2 f(x)}{dx^2} = -\frac{1}{g}\frac{d^2 g(z)}{dz^2} - k_0^2 \tag{3.11}$$

The left-hand side of equation (3.11) depends only on x and the right-hand side only on z. A change in x cannot affect the right-hand side and vice versa. Because equation (3.11) is valid for all values of x and z, it is necessary that both sides are equal to some (complex) constant, which will be denoted by $-b^2$. Without limiting the generality, the convention can clearly be made that $-\pi/2 < \arg b \leqslant \pi/2$. Thus equation (3.11) is resolved into two equations:

$$\frac{d^2 f(x)}{dx^2} + b^2 f(x) = 0 \tag{3.12}$$

and

$$\frac{d^2 g(z)}{dz^2} + (k_0^2 - b^2)g(z) = 0 \,. \tag{3.13}$$

If b differs from zero, the general solution of equation (3.12) is

$$f(x) = C_1 e^{ibx} + C_2 e^{-ibx} . \tag{3.14}$$

For $b = 0$ the general solution is

$$f(x) = C_0 x + C_0' . \tag{3.15}$$

Here C_1, C_2, C_0 and C_0' are constants. Let us not allow E_y of formula (3.10) to grow to infinity as x goes to $+\infty$ or $-\infty$. C_0 must thus be zero, and if the imaginary part of b does not vanish, C_1 and C_2 must be zero. In other words, only real, and non-negative (because $-\pi/2 < \arg b \leqslant \pi/2$) values of b are acceptable and all solutions of equation (3.12) are linear combinations of e^{ibx} and e^{-ibx}. The quantity $\pm b$ is actually the argument of the Fourier transforms of the source current and of the fields with respect to the x-space-coordinate (see Section A.4).

Let us use the following notation:

$$\kappa_0^2 = b^2 - k_0^2 \tag{3.16}$$

and assume that

$$-\frac{\pi}{2} < \arg \kappa_0 \leqslant \frac{\pi}{2} . \tag{3.17}$$

Because b and k_0 are real, it follows from condition (3.17) that

$$\arg \kappa_0 = \begin{cases} 0 , & b > k_0 \\ \frac{\pi}{2} , & b < k_0 \end{cases} \tag{3.18}$$

The solution of equation (3.13) can be expressed as

$$g(z) = A_1 e^{\kappa_0 z} + A_2 e^{-\kappa_0 z} \tag{3.19}$$

for $\kappa_0 \neq 0$ and as

$$g(z) = A_0 z + A_0' \tag{3.20}$$

for $\kappa_0 = 0$. Formula (3.18) shows that for $b > k_0$ the function $e^{-\kappa_0 z}$ represents a field which approaches infinity as z approaches minus infinity, *i.e.* when the point of observation moves far upwards from the earth's surface. On the contrary, the function $e^{\kappa_0 z}$ satisfies $\lim_{z \to -\infty} e^{\kappa_0 z} = 0$. For $b < k_0$, κ_0 can be written as $i\alpha$ where

α is real and positive, *i.e.* $e^{i\omega t}e^{\pm\kappa_0 z} = e^{i(\omega t \pm \alpha z)}$. This means that the function $e^{-\kappa_0 z}$ belongs to a field whose phase propagation with respect to z occurs in the $+z$-direction (downwards), while the propagation direction associated with $e^{\kappa_0 z}$ is upwards. Thus it seems reasonable that the z-dependence of the secondary field shall be represented by $e^{\kappa_0 z}$ both for $b > k_0$ and $b < k_0$.

If b equals k_0, the solution for the z-dependence is given by equation (3.20). The function proportional to z could be rejected by stating that it grows infinitely large as z approaches minus infinity, which is unphysical. On the other hand, an integral over b will be composed later, and the value of the integral does not change although the values of the integrand are changed at separate points (see Section A.4 and the discussion after formula (2.66) in Section 2.5). Because of this it could be argued that there is no reason to discuss the case $b = k_0$. We shall, however, now avoid the special case $\kappa_0 = 0$ entirely by assuming that k_0 has a (small) negative imaginary part. Physically this would be achieved by providing the half space $z < 0$ (the air) with a (slight) conductivity σ_0, and it is assumed to be the case in this discussion. The conductivity is presumed to be constant with respect to time and space. (A similar assumption of a very small conductivity has also been made for example by SOMMERFELD, 1959, p. 160, in the treatment of a different electromagnetic problem.) In reality the conductivity of the air differs from zero and is of the order of $10^{-14}\Omega^{-1}\text{m}^{-1}$ near the earth's surface (ISRAËL, 1971, pp. 95 and 249). If a non-zero conductivity is given to a medium the real part of its propagation constant also changes as can be seen from equation (B.46). The discussion that follows and in which expressions for the electromagnetic field on the earth's surface are derived, is in principle valid for any values of the permittivity and of the permeability of the air. We, however, assign the correct free space values ϵ_0 and μ_0 to these parameters. The assumption made about the conductivity of the air and the definition of κ_0 by formulae (3.16) and (3.17) imply that for all possible values of b both the real and the imaginary parts of κ_0 are positive. Consequently the function $e^{\kappa_0 z}$, with the time-dependence $e^{i\omega t}$, is connected with upward phase propagation and it approaches zero as z approaches minus infinity. On the other hand the function $e^{-\kappa_0 z}$ indicates downward propagation and approaches infinity as z approaches minus infinity. Therefore the former function only is accepted, which is exactly the same conclusion as above with a real value of k_0 and $b \neq k_0$.

The conductivity of the air also modifies the primary field discussed in Section 3.1. Referring to Section B.8 it is, however, evident that the expressions (3.5) and (3.6) are formally valid for the electromagnetic field produced by an external (true) current (3.1) in a conducting medium, if k_0 is assumed to represent the correct propagation constant including the conductivity. (The primary charge density that could in principle exist, but is assumed to be zero, would now not be time-independent, but similar to formula (B.27).)

If the argument $k_0\sqrt{x^2 + (z + h)^2} = k_0 r_0$ of the Hankel functions appearing in the expressions of the primary field is so large that the asymptotic formula (A.59) can be used for $H_0^{(2)}$ and $H_1^{(2)}$, both the primary electric field and the primary magnetic field are proportional to $(k_0 r_0)^{-1/2} \cdot e^{i(\omega t - k_0 r_0)}$. Thus owing to formula (B.42) being valid for k_0, the phase propagation of the primary field occurs outward from the source current, and the conductivity of the air causes exponential damping in the same direction. Formulae (3.7) and (3.8) show that in practice, *i.e.* for reasonable values of $k_0 r_0$, the primary electric field is only affected by the conductivity of the air.

The general solution of equation (3.9) associated with a separation constant b^2 is a linear combination of the products of all possible functions $f(x)$ and $g(z)$, multiplied by the time factor $e^{i\omega t}$, *i.e.*

$$E_y(x, z, t) = (D_{b0} e^{ibx} e^{\kappa_0 z} + F_{b0} e^{-ibx} e^{\kappa_0 z}) e^{i\omega t}. \tag{3.21}$$

The coefficients D_{b0} and F_{b0} are integration constants. In order to get a still more general solution for E_y it is necessary to sum (integrate) over all possible values of b (*cf.* MORSE and FESHBACH, 1953, p. 498):

$$E_y(x, z, t) = e^{i\omega t} \int_0^{\infty} (D_0(b) e^{ibx} + F_0(b) e^{-ibx}) e^{\kappa_0 z} db \ , \tag{3.22}$$

in which the coefficients D_{b0} and F_{b0} have been replaced by $D_0(b)db$ and $F_0(b)db$, respectively. The mathematical treatment becomes more convenient by defining the unknown »integration constant» function $D_0(b)$ for non-positive values of b by the formulae: $D_0(b) = F_0(-b)$ for $b < 0$, and $D_0(b{=}0) = (1/2)(D_0'(b{=}0) + F_0(b{=}0))$ where $D_0'(b{=}0)$ denotes the original value of $D_0(b)$ at $b = 0$. The component E_y can then be expressed as

$$E_y(x, z, t) = e^{i\omega t} \int_{-\infty}^{\infty} D_0(b) e^{\kappa_0 z} e^{ibx} db \ . \tag{3.23}$$

The quantity κ_0 is defined by formulae (3.16) and (3.17) for negative values of b as well.

The components B_x and B_z are obtained from equation (3.23) with the use of formula (B.37):

$$\bar{B} = -\frac{1}{i\omega} \nabla \times \bar{E} = \frac{1}{i\omega} \frac{\partial E_y}{\partial z} \hat{e}_x + \frac{1}{i\omega} \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \hat{e}_y - \frac{1}{i\omega} \frac{\partial E_y}{\partial x} \hat{e}_z \ , \tag{3.24}$$

and hence

$$B_x(x, z, t) = \frac{e^{i\omega t}}{i\omega} \int_{-\infty}^{\infty} \kappa_0 D_0(b) e^{\kappa_0 z} e^{ibx} db \tag{3.25}$$

and

$$B_z(x, z, t) = -\frac{e^{i\omega t}}{\omega} \int_{-\infty}^{\infty} b D_0(b) e^{\kappa_0 z} e^{ibx} db \,, \tag{3.26}$$

Utilizing the discussion of MORSE and FESHBACH, 1953, pp. 497–498 the conclusion may probably be made that all physical solutions for the secondary field components E_y, B_x and B_z in the air are included in formulae (3.23), (3.25) and (3.26).

Expressing the last Maxwell equation (B.38), with $\sigma = \sigma_0$, $\epsilon = \epsilon_0$ and $\mu = \mu_0$, in component form similar to formula (3.24) we observe that the components E_y, B_x, B_z and the components E_x, E_z, B_y constitute two independent sets of equations. Because the primary field has only the components E_y, B_x and B_z, let us not include the components E_x, E_z and B_y in the treatment. The latter could be solved exactly analogously to the calculation of E_y, B_x and B_z above.

It is mentioned in Section B.5 that the validity of Maxwell's equations (B.35)–(B.38) does not result from the validity of the wave equations (B.43) and (B.44). Because the calculation above included the use of the wave equation for the electric field, it should now be checked that the electromagnetic field (E_y, B_x, B_z) satisfies equations (B.35)–(B.38). Equation (B.35) is trivial because the derivative $\partial/\partial y$ is zero, the calculation of the magnetic field (formula (3.24)) involves the validity of equation (B.37), the latter implies the validity of equation (B.36), and finally it is easy to show the equation (B.38) results from the wave equation (B.43) and equations (B.35) and (B.37) (*cf.* Section B.6).

The electromagnetic field within the j^{th} layer of the earth satisfies Maxwell's equations (B.35)–(B.38) with $\sigma = \sigma_j$, $\epsilon = \epsilon_j$ and $\mu = \mu_j$ (j = 1,..., n). The solution is obtained with the same procedure as the solution for the secondary field above. However, the z-dependence of the separated solution for E_y is now different. Let us make the reasonable assumption that the conductivity of the earth is everywhere non-zero. Then the special cases are excluded where one or more of the quantities κ_j, corresponding to κ_0 and defined by formulae

$$\kappa_j^2 = b^2 - k_j^2 \tag{3.27}$$

and

$$-\frac{\pi}{2} < \arg \kappa_j \leqslant \frac{\pi}{2}, \tag{3.28}$$

are zero. The parameter k_j is the propagation constant of the j^{th} layer. It results from formulae (3.27) and (3.28) and from the non-zero conductivities $\sigma_j (j=1,...,n)$ that both the real and the imaginary parts of all factors κ_j are positive. To avoid infinite growth only the solution $e^{-\kappa_j z}$ is accepted in the undermost layer, *i.e.* when j = n. In all other layers the z-dependence of E_y includes both $e^{\kappa_j z}$ and $e^{-\kappa_j z}$. Thus similarly to formula (3.23) the y-component of the electric field in the earth has the expression

$$E_y(x, z, t) = e^{i\omega t} \int_{-\infty}^{\infty} (D_j(b)e^{\kappa_j z} + G_j(b)e^{-\kappa_j z})e^{ibx} db \tag{3.29}$$

for j = 1,..., n−1, and

$$E_y(x, z, t) = e^{i\omega t} \int_{-\infty}^{\infty} G_n(b)e^{-\kappa_n z}e^{ibx} db \tag{3.30}$$

for the lowest layer j = n. $D_j(b)$ and $G_j(b)$ are unknown »integration constant» functions. Equation $\bar{B} = -(1/i\omega)\cdot\nabla\times\bar{E}$ now yields that

$$B_x(x, z, t) = \frac{e^{i\omega t}}{i\omega} \int_{-\infty}^{\infty} \kappa_j(D_j(b)e^{\kappa_j z} - G_j(b)e^{-\kappa_j z})e^{ibx} db \tag{3.31}$$

and

$$B_z(x, z, t) = -\frac{e^{i\omega t}}{\omega} \int_{-\infty}^{\infty} b(D_j(b)e^{\kappa_j z} + G_j(b)e^{-\kappa_j z})e^{ibx} db \tag{3.32}$$

for j = 1,..., n − 1, and

$$B_x(x, z, t) = -\frac{e^{i\omega t}}{i\omega} \int_{-\infty}^{\infty} \kappa_n G_n(b)e^{-\kappa_n z}e^{ibx} db \tag{3.33}$$

and

$$B_z(x, z, t) = -\frac{e^{i\omega t}}{\omega} \int_{-\infty}^{\infty} b\, G_n(b)e^{-\kappa_n z}e^{ibx} db \tag{3.34}$$

for j = n.

Referring to MORSE and FESHBACH, 1953, pp. 497–498, it can again be stated that all physical solutions for the field components E_y, B_x and B_z are obviously included in equations (3.29)–(3.34). As in the case of the secondary field in the air the other three field components are not included in the treatment. The validity of Maxwell's equations with formulae (3.29)–(3.34) is evident. The present treatment, as the discussion in Chapter 2, implies that no charges exist, even on the surfaces of discontinuity (see Sections B.5 and B.7).

Before starting to consider the boundary conditions let us discuss the space- and time-dependent functions appearing in the above expressions for the field. Since both the real and the imaginary parts of all factors κ_j (j = 0, 1, ..., n) are positive and b is real, the function $e^{i(\omega t+bx)+\kappa_j z}$ involves phase propagation and attenuation in the negative z-direction and with respect to x no attenuation occurs. For $b = 0$ the phase propagation with x disappears. Similarly the function $e^{i(\omega t+bx)-\kappa_j z}$ is associated with phase propagation and attenuation in the positive z-direction. If the conductivity of the medium in question were zero, either phase propagation or attenuation, or both when $\kappa_j = 0$, would vanish with respect to z. The solution proportional to z (equation (3.20)) involves no phase propagation. So, taking into account the discussion above of the behaviour of the primary field for large values of $k_0 r_0$, no case where the directions of phase propagation and attenuation with respect to a space-coordinate are opposite appears in the present discussion. Such cases will be discussed in Chapter 4.

Here we have discussed the phase propagation, but it is the direction of energy flow, *i.e.* the Poynting vector, that is more significant (see the end of Section B.9). The Poynting vector for transverse magnetic cylindrical electromagnetic fields, which are independent of the cylindrical φ-coordinate, is discussed in Appendix C. The conclusions observed there are also applicable to similar planar fields. However, the integrands, associated with a value of b, of the secondary field in the air or of the field in the earth do not compose fields of this type. So the discussion of Appendix C is not available, and a new treatment of the Poynting vector would be needed. Such a treatment would show that the energy of the field associated with a fixed value of b flows in the direction of phase propagation on average. However, in the investigation of the Poynting vector of a field represented as a sum (or integral) the individual terms of the sum cannot be considered separately, as otherwise the cross terms in the Poynting vector vanish. Finally, concerning the primary field of this chapter the discussion of Appendix C is applicable with the longitudinal propagation constant equal to zero.

According to Section B.7 the tangential components E_y and $H_x = B_x/\mu$ are continuous at the boundary surfaces $z = 0$, $z = z_1 = h_1$, $z = z_2 = h_1 + h_2, \ldots$, $z = z_{n-1} = h_1 + h_2 + \ldots + h_{n-1}$. Therefore equations (3.5), (3.6), (3.23), (3.25), (3.29), (3.30), (3.31) and (3.33) yield that

$$-\frac{\omega\mu_0 J}{4} H_0^{(2)}(k_0\sqrt{x^2+h^2}) + \int_{-\infty}^{\infty} D_0(b)e^{ibx}\,db = \int_{-\infty}^{\infty} (D_1(b)+G_1(b))e^{ibx}\,db \tag{3.35}$$

and

$$-\frac{ik_0 hJ}{4\sqrt{x^2+h^2}} H_1^{(2)}(k_0\sqrt{x^2+h^2}) + \frac{1}{i\omega\mu_0}\int_{-\infty}^{\infty} \kappa_0 D_0(b)e^{ibx}\,db = \tag{3.36}$$

$$= \frac{1}{i\omega\mu_1}\int_{-\infty}^{\infty} \kappa_1 (D_1(b)-G_1(b))e^{ibx}db$$

for $z = 0$,

$$\int_{-\infty}^{\infty} (D_j(b)e^{\kappa_j z_j} + G_j(b)e^{-\kappa_j z_j})e^{ibx}db = \int_{-\infty}^{\infty} (D_{j+1}(b)e^{\kappa_{j+1} z_j} + G_{j+1}(b)e^{-\kappa_{j+1} z_j})e^{ibx}db \tag{3.37}$$

and

$$\frac{1}{i\omega\mu_j}\int_{-\infty}^{\infty} \kappa_j(D_j(b)e^{\kappa_j z_j} - G_j(b)e^{-\kappa_j z_j})e^{ibx}db = \frac{1}{i\omega\mu_{j+1}}\int_{-\infty}^{\infty} \kappa_{j+1}(D_{j+1}(b)e^{\kappa_{j+1} z_j} + \tag{3.38}$$

$$- G_{j+1}(b)e^{-\kappa_{j+1} z_j})e^{ibx}db$$

for $z = z_j$ where $j = 1,\ldots, n-2$, and

$$\int_{-\infty}^{\infty} (D_{n-1}(b)e^{\kappa_{n-1} z_{n-1}} + G_{n-1}(b)e^{-\kappa_{n-1} z_{n-1}})e^{ibx}db = \int_{-\infty}^{\infty} G_n(b)e^{-\kappa_n z_{n-1}}\,e^{ibx}db \tag{3.39}$$

and

$$\frac{1}{i\omega\mu_{n-1}}\int_{-\infty}^{\infty} \kappa_{n-1}(D_{n-1}(b)e^{\kappa_{n-1} z_{n-1}} - G_{n-1}(b)e^{-\kappa_{n-1} z_{n-1}})e^{ibx}db = \tag{3.40}$$

$$= -\frac{1}{i\omega\mu_n}\int_{-\infty}^{\infty} \kappa_n G_n(b)e^{-\kappa_n z_{n-1}}e^{ibx}db$$

for $z = z_{n-1}$. Equations (3.35)–(3.40) have to be valid for all values of x. The common time factor $e^{i\omega t}$ has been divided out from these equations. Notice that the exact formulae (3.5) and (3.6) are used for the primary field, not the approximative (though valid) formulae (3.7) and (3.8).

Utilizing formulae (A.4) and (A.6) it follows from equations (3.37)–(3.40) that

$$D_j(b)e^{\kappa_j z_j} + G_j(b)e^{-\kappa_j z_j} - D_{j+1}(b)e^{\kappa_{j+1} z_j} - G_{j+1}(b)e^{-\kappa_{j+1} z_j} = 0\,, \tag{3.41}$$

and

$$\frac{\kappa_j}{\mu_j} D_j(b)e^{\kappa_j z_j} - \frac{\kappa_j}{\mu_j} G_j(b)e^{-\kappa_j z_j} - \frac{\kappa_{j+1}}{\mu_{j+1}} D_{j+1}(b)e^{\kappa_{j+1} z_j} + \frac{\kappa_{j+1}}{\mu_{j+1}} G_{j+1}(b)e^{-\kappa_{j+1} z_j} = 0 \tag{3.42}$$

(j = 1,..., n − 2), and

$$D_{n-1}(b)e^{\kappa_{n-1} z_{n-1}} + G_{n-1}(b)e^{-\kappa_{n-1} z_{n-1}} - G_n(b)e^{-\kappa_n z_{n-1}} = 0 \tag{3.43}$$

and

$$\frac{\kappa_{n-1}}{\mu_{n-1}} D_{n-1}(b)e^{\kappa_{n-1} z_{n-1}} - \frac{\kappa_{n-1}}{\mu_{n-1}} G_{n-1}(b)e^{-\kappa_{n-1} z_{n-1}} + \frac{\kappa_n}{\mu_n} G_n(b)e^{-\kappa_n z_{n-1}} = 0. \tag{3.44}$$

These equations are valid for all real values of b, and for each b they constitute a set of 2n−2 linear equations, which involve 2n−1 unknown integration constants. So one of the unknowns, for instance $G_1(b)$, can be regarded as known and all of them obtained in the forms

$$D_j(b) = \alpha_j(b)G_1(b) \tag{3.45}$$

and

$$G_j(b) = \beta_j(b)G_1(b) \tag{3.46}$$

(j = 1,..., n). The coefficients $\alpha_j(b)$ and $\beta_j(b)$ depend on the electromagnetic properties and the thicknesses of the layers of the earth. ($\alpha_n(b) = 0$ and $\beta_1(b) = 1$.)

It follows from formulae (3.29), (3.31) and (3.45) that the tangential components of the electromagnetic field at the earth's surface on its lower side are

$$E_y(x, z = 0, t) = \int_{-\infty}^{\infty} E_y(b, x, t)db \tag{3.47}$$

and

$$B_x(x, z = 0, t) = \int_{-\infty}^{\infty} B_x(b, x, t)db \tag{3.48}$$

where

$$E_y(b, x, t) = G_1(b)(\alpha_1(b) + 1)e^{ibx}e^{i\omega t} \tag{3.49}$$

and

$$B_x(b,x,t) = G_1(b)\,\frac{\kappa_1}{i\omega}\,(\alpha_1(b) - 1)e^{ibx}e^{i\omega t}(= \mu_1 H_x(b,x,t))\,. \tag{3.50}$$

Let us define the surface impedance $Z(b)$ at the earth's surface by

$$Z(b) = -\frac{E_y(b,x,t)}{H_x(b,x,t)} = -\mu_1\,\frac{E_y(b,x,t)}{B_x(b,x,t)} = \frac{i\omega\mu_1}{\kappa_1}\,\frac{1+\alpha_1(b)}{1-\alpha_1(b)}\,. \tag{3.51}$$

Because all quantities κ_j are even with respect to b, equations (3.41)–(3.44) are identical for $+b$ and $-b$, and so $\alpha_j(b) = \alpha_j(-b)$ and $\beta_j(b) = \beta_j(-b)$ and further

$$Z(b) = Z(-b)\,. \tag{3.52}$$

SRIVASTAVA, 1965, and ALBERTSON and VAN BAELEN, 1970, referring to Srivastava, give this surface impedance a more complete formula in the sense that they do not introduce any quantity, like $\alpha_1(b)$ above, which is not expressed explicitly in terms of the parameters of the earth. But their formula assumes that the permeability in every layer equals the free space permeability, and that the displacement currents can be neglected, *i.e.* $\sigma_j \gg \omega\epsilon_j$. In Srivastava's analysis the ratios $-E_y/H_x$ and E_x/H_y are both equal to $Z(b)$, denoted by $Z(0)$. However, if the components E_x, E_z and B_y were treated in the same way as E_y, B_x and B_z above in the present discussion, the ratio $E_x(b,x,t)/H_y(b,x,t)$, denoted here by $Z'(b)$, would have the following expression:

$$Z'(b) = \frac{\kappa_1}{\sigma_1 + i\omega\epsilon_1}\,\frac{1-\alpha_1'(b)}{1+\alpha_1'(b)}\,, \tag{3.53}$$

where $\alpha_1'(b)$ is obtained from $\alpha_1(b)$ by replacing the quantities $-i\omega\mu_j$ by $\sigma_j + i\omega\epsilon_j$ ($j = 1,\ldots, n$). This substitution, which is a consequence of the symmetry of the field vectors $\bar{E}$ and $\bar{H}$ in Maxwell's equations (B.35)–(B.38) ($\bar{B} = \mu\bar{H}$) and in the boundary conditions, was also utilized when concluding the coefficient $\kappa_1/(\sigma_1 + i\omega\epsilon_1)$ in equation (3.53) from the factor $i\omega\mu_1/\kappa_1$ in formula (3.51). Since $Z(b)$ and $Z'(b)$ are not equal either when $\mu_j = \mu_0$ and $\sigma_j \gg \omega\epsilon_j$, the present discussion apparently contradicts Srivastava's treatment, which might even be considered more general thanks to the possibility of y-dependence. The discrepancy is, however, due to the difference in the assumptions: Srivastava assumes that E_z for the value of b in question is zero within the earth. In the present analysis the starting point is that the derivatives $\partial/\partial y$ vanish. As seen, these two different assumptions yield the same value for the ratio $-E_y/H_x$ but different values for E_x/H_y. If in the present treatment b is zero, which means that only z-space-dependence occurs (plane waves), E_z is necessarily zero (see Section B.6), and so $Z'(b)$ must equal $Z(b)$. The fact that the case is really so is seen from Chapter 2 and may also be concluded with the substitution $b = 0$ into formulae (3.51) and (3.53).

In the same way that equations (3.41)–(3.44) were obtained from formulae (3.37)–(3.40) it follows from equations (3.35) and (3.36) that

$$D_0(b) - (1 + \alpha_1(b))G_1(b) = \frac{i\omega\mu_0 J}{4\pi} \frac{e^{-\kappa_0 h}}{\kappa_0} \tag{3.54}$$

and

$$\frac{\kappa_0}{\mu_0} D_0(b) + \frac{\kappa_1}{\mu_1}(1 - \alpha_1(b))G_1(b) = -\frac{i\omega J}{4\pi} e^{-\kappa_0 h} \tag{3.55}$$

where formulae (A.72), (A.75) and (3.45) have been employed. As mentioned, the small negative imaginary part of k_0 ensures that $\kappa_0 = \sqrt{b^2 - k_0^2}$ differs from zero for all values of b, and thus the right-hand side of formula (3.54) remains finite and well defined. The use of equations (A.72) and (A.75) does not demand a positive real part for ik_0, but the case $Re(ik_0) = 0$, *i.e.* $Im k_0 = 0$, is also allowed. (k_0 must, however, differ from zero.) The quantities $D_0(b)$ and $G_1(b)$ can be obtained from equations (3.54) and (3.55). The latter will not be used in this discussion, and the former has the expression

$$D_0(b) = -\frac{i\omega\mu_0 J e^{-\kappa_0 h}}{4\pi\kappa_0} \frac{\kappa_0 Z(b) - i\omega\mu_0}{\kappa_0 Z(b) + i\omega\mu_0}, \tag{3.56}$$

where formula (3.51) has also been utilized.

We obtain from formulae (3.5), (3.6), (3.23), (3.25), (3.26), (3.56), (A.40), (A.74) and (A.76) the following electromagnetic field components at the earth's surface on its upper side:

$$\begin{aligned} E_{My}(x, t) = &-\frac{\omega\mu_0 J e^{i\omega t}}{4} H_0^{(2)}(k_0\sqrt{x^2 + h^2}) \\ &-\frac{i\omega\mu_0 J e^{i\omega t}}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-\kappa_0 h}}{\kappa_0} \frac{\kappa_0 Z(b) - i\omega\mu_0}{\kappa_0 Z(b) + i\omega\mu_0} e^{ibx} db \\ = &-\frac{i\omega\mu_0 J e^{i\omega t}}{2\pi} \int_{-\infty}^{\infty} \frac{Z(b) e^{-\kappa_0 h}}{\kappa_0 Z(b) + i\omega\mu_0} e^{ibx} db \end{aligned} \tag{3.57}$$

$$B_{Mx}(x,t) = -\frac{i\mu_0 k_0 h J e^{i\omega t}}{4\sqrt{x^2+h^2}} H_1^{(2)}(k_0\sqrt{x^2+h^2}) \tag{3.58}$$

$$-\frac{\mu_0 J e^{i\omega t}}{4\pi} \int_{-\infty}^{\infty} e^{-\kappa_0 h} \frac{\kappa_0 Z(b) - i\omega\mu_0}{\kappa_0 Z(b) + i\omega\mu_0} e^{ibx} db$$

$$= \frac{i\omega\mu_0^2 J e^{i\omega t}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\kappa_0 h}}{\kappa_0 Z(b) + i\omega\mu_0} e^{ibx} db .$$

and

$$B_{Mz}(x,t) = \frac{i\mu_0 k_0 x J e^{i\omega t}}{4\sqrt{x^2+h^2}} H_1^{(2)}(k_0\sqrt{x^2+h^2}) \tag{3.59}$$

$$+ \frac{i\mu_0 J e^{i\omega t}}{4\pi} \int_{-\infty}^{\infty} \frac{b e^{-\kappa_0 h}}{\kappa_0} \frac{\kappa_0 Z(b) - i\omega\mu_0}{\kappa_0 Z(b) + i\omega\mu_0} e^{ibx} db$$

$$= \frac{i\mu_0 J e^{i\omega t}}{2\pi} \int_{-\infty}^{\infty} \frac{b Z(b) e^{-\kappa_0 h}}{\kappa_0 Z(b) + i\omega\mu_0} e^{ibx} db$$

When deriving formula (3.59), the derivative of equation (A.74) with respect to u was used changing the order of the derivative and of the integral. This can be shown to be acceptable. The treatment of the components E_x, E_z and B_y was neglected above. As indicated, the discussion about them would be completely independent of the treatment of E_y, B_x, B_z and they would not be coupled to the primary field. Hence, we conclude that E_x, E_z and B_y are zero everywhere, and so formulae (3.57)–(3.59) really represent the whole electromagnetic field.

Equations (3.57) and (3.58) show that the ratio of the integrands of E_{My} and of B_{Mx} including the coefficient before the integration sign is equal to $-Z(b)/\mu_0$. Therefore the surface impedance, whose definition (3.51) was connected with the field inside the earth, is for each b equal to the negative ratio of the perpendicular y- and x-components of the electric field and of the magnetic field intensity, respectively, at the earth's surface on its upper side. This is not surprising, since according to formulae (3.54) and (3.55), *i.e.* according to the continuity E_y and H_x, the quantities $E_y(b,x,t)$ and $H_x(b,x,t) = (1/\mu_1)B_x(b,x,t)$ given by equations (3.49) and (3.50) are equal to the integrands of the b-integrals representing E_y and $H_x = (1/\mu_0)B_x$ at the earth's surface on its upper side. These integrals, like all b-integrals in this work, are Fourier integral representations in x.

The surface impedance $Z(b)$ only depends on the properties of the earth, and it is obtained by assuming a harmonic time-dependence and by setting the derivatives $\partial/\partial y$ equal to zero. (According to the discussion after formula (3.52) it is obvious that the same surface impedance would have been obtained without the presumption $\partial/\partial y = 0$, if E_z for the value of b in question is assumed to be zero within the earth.)

Utilizing equations (3.16), (3.52) and (A.29) formulae (3.57)–(3.59) describing the electromagnetic field on the earth's surface can be rewritten as

$$E_{My}(x, t) = -\frac{i\omega\mu_0 J e^{i\omega t}}{\pi} \int_0^\infty \frac{Z(b) e^{-\kappa_0 h}}{\kappa_0 Z(b) + i\omega\mu_0} \cos bx db\,, \tag{3.60}$$

$$B_{Mx}(x, t) = \frac{i\omega\mu_0^2 J e^{i\omega t}}{\pi} \int_0^\infty \frac{e^{-\kappa_0 h}}{\kappa_0 Z(b) + i\omega\mu_0} \cos bx db \tag{3.61}$$

and

$$B_{Mz}(x, t) = -\frac{\mu_0 J e^{i\omega t}}{\pi} \int_0^\infty \frac{b Z(b) e^{-\kappa_0 h}}{\kappa_0 Z(b) + i\omega\mu_0} \sin bx db \tag{3.62}$$

These equations show that E_{My} and B_{Mx} are even and B_{Mz} odd in x, so that in this respect the total electromagnetic field has the same properties as the primary field (equations (3.5) and (3.6)). It was assumed above that k_0 has a non-zero negative imaginary part, *i.e.* that the air is slightly conducting. The conductivity can be arbitrarily small (but positive), and so equations (3.60)–(3.61) express the electromagnetic field on the earth's surface in the case where the properties of the upper half-space are arbitrarily near those of free space. Because »physics is continuous», it seems evident that in the limit where the conductivity of the air approaches zero formulae (3.60)–(3.62) give the electromagnetic field for free space, and this limit is obviously achieved with the substitution $k_0 = \omega\sqrt{\mu_0\epsilon_0}$.

If equations (3.60) and (3.61) are compared with the results obtained by Albertson and Van Baelen, 1970, a difference is observed. These authors have the variable of integration in place of κ_0 in equations (3.60) and (3.61). Albertson and Van Baelen assume in their discussion that the upper half-space is free space without any conductivity, but this is not, of course, the reason for the difference. As mentioned in Section 3.1, the propagation constant k_0 is small for frequencies significant in connection with geomagnetic variations. This statement is not made invalid by the small conductivity of the upper half space. Hence the quantity $\kappa_0 = \sqrt{b^2 - k_0^2}$ can be approximated by b for all but very small values of b. The integration in formulae (3.60) and (3.61), however, starts from $b = 0$. Therefore it is not in principle allowed to replace κ_0 by b in them and so make them similar to

Albertson's and Van Baelen's equations. The approximation included if this »forbidden» substitution is made will be discussed later in a special case.

As indicated, the treatment of the functions $H_0^{(2)}(k_0\sqrt{x^2+h^2})$, $(-k_0h/\sqrt{x^2+h^2})$. $H_1^{(2)}(k_0\sqrt{x^2+h^2})$ and $(-k_0x/\sqrt{x^2+h^2})\,H_1^{(2)}(k_0\sqrt{x^2+h^2})$ in the derivation of equations (3.54) and (3.55) and in formulae (3.57)–(3.59) involves the expression of the primary electromagnetic field at the earth's surface as a Fourier integral with respect to x (see formulas (A.72) and (A.74)–(A.76)). The primary source current (3.1) also has a Fourier integral representation with respect to x, and the treatment of the electromagnetic field connected with a particular value of b actually means the consideration of the influence of the corresponding Fourier component of the source current (*cf.* the principle of superposition in Section B.10, see also Chapter 5). Hence every partial field associated with a certain value of b has a clear physical meaning, though b (or $-b^2$) appeared in the discussion as a purely mathematical separation constant of equation (3.11). It was therefore quite reasonable to demand physically acceptable behavior of every partial electromagnetic field, as was done above.

The Fourier integral representation with respect to x for the primary electromagnetic field is also implicit in ALBERTSON's and VAN BAELEN's, 1970, calculations. They assume that the approximate expression (3.8) for the primary magnetic field is valid, which allows this field to be expressed as the negative gradient of a magnetic scalar potential. The Fourier and inverse Fourier (sine) transforms of this potential are given explicitly by these authors.

As mentioned in Section 3.1, formulae (3.7) and (3.8) can be used for values of ω, x and $z+h$ feasible in connection with geomagnetic induction. But the use of formula (3.8) in a Fourier transform with respect to x, where integration goes to infinity, in principle requires that ω is zero, which in turn means that the electric field (3.7) vanishes. The error, which is caused for $\omega \neq 0$ in Albertson's and Van Baelen's discussion, is equivalent to the substitution of b for κ_0 in the discussion above. If we try to integrate the Fourier integral representation of the primary electric field included implicitly in Albertson's and Van Baelen's treatment, we find difficulties, because the integral does not converge, but has an infinite value. This integral is, however, multiplied by a factor proportional to ω, which, as stated, should in principle be zero. The result is then exactly the same as equation (3.7) with $\omega = 0$, as can be expected. Similar »inaccurate» Fourier integral expressions for the primary magnetic field or its scalar potential, which in any case yield the magnetic field very accurately for reasonable values of ω, x and $z+h$, are also represented for instance by PRICE, 1962, HERMANCE and PELTIER, 1970, and HERMANCE, 1978.

An expression for the potential difference between two points $P_1 = (x_1, y_1, 0)$ and $P_2 = (x_2, y_2, 0)$ at the earth's surface is obtained using the convention made in Chapter 1 and equation (3.60). It is

$$U_{P_1 P_2}(t) = \int_{P_1 \, \text{straight line}}^{P_2} \bar{E}_M \cdot d\bar{l} = \int_{P_1 \, s.l.}^{P_2} E_{My}(x, t) dy \tag{3.63}$$

$$= -\frac{i\omega\mu_0 (y_2 - y_1) J e^{i\omega t}}{\pi(x_2 - x_1)} \int_0^\infty \frac{Z(b) e^{-\kappa_0 h}}{b(\kappa_0 Z(b) + i\omega\mu_0)} (\sin bx_2 - \sin bx_1) db \, .$$

As in Section 2.2, this equation gives the potential drop from P_1 to P_2. If x_1 and x_2 are equal, the potential difference $U_{P_1 P_2}$ is simply

$$U_{P_1 P_2}(t) = (y_2 - y_1) E_{My}(x, t) \tag{3.64}$$

$$= -\frac{i\omega\mu_0 (y_2 - y_1) J e^{i\omega t}}{\pi} \int_0^\infty \frac{Z(b) e^{-\kappa_0 h}}{\kappa_0 Z(b) + i\omega\mu_0} \cos bx db \, ,$$

where x is the common value of x_1 and x_2. The primary current $Je^{i\omega t}$ can be solved from equation (3.61) or equation (3.62) as a function of a magnetic field component on the earth's surface. Hence the potential difference $U_{P_1 P_2}$ is expressible in terms of the magnetic field (*cf.* equation (2.36)). The physical electric field and the physical geomagnetic variation on the earth's surface and the physical potential difference between P_1 and P_2 are, according to Section A.1, expressed by the real parts of equations (3.60)–(3.63).

3.3 Induction in a homogeneous earth

If the earth is assumed to be homogeneous (with σ, ϵ, μ, k and κ) the impedance $Z(b)$ is

$$Z(b) = \frac{i\omega\mu}{\kappa} \tag{3.65}$$

as can be seen from the definition of $Z(b)$ (3.51) and equations (3.30) and (3.33), which describe the field in the only layer. Substituting equation (3.65) in equations (3.60)–(3.62) we obtain:

$$E_{My}(x, t) = -\frac{i\omega\mu_0 J e^{i\omega t}}{\pi} \int_0^\infty \frac{e^{-\kappa_0 h}}{\kappa_0 + \kappa} \cos bx db \, , \tag{3.66}$$

$$B_{Mx}(x,t) = \frac{\mu_0 J e^{i\omega t}}{\pi} \int_0^\infty \frac{\kappa e^{-\kappa_0 h}}{\kappa_0 + \kappa} \cos bx db \tag{3.67}$$

and

$$B_{Mz}(x,t) = -\frac{\mu_0 J e^{i\omega t}}{\pi} \int_0^\infty \frac{b e^{-\kappa_0 h}}{\kappa_0 + \kappa} \sin bx db . \tag{3.68}$$

The potential difference given by equation (3.63) has the following formula for a homogeneous earth:

$$U_{P_1P_2}(t) = -\frac{i\omega\mu_0(y_2 - y_1)Je^{i\omega t}}{\pi(x_2 - x_1)} \int_0^\infty \frac{e^{-\kappa_0 h}}{b(\kappa_0 + \kappa)} (\sin bx_2 - \sin bx_1) db . \tag{3.69}$$

The expressions (3.66)–(3.69) presume that μ and μ_0 are equal. Unless this assumption were made, $B_{Mx}(x, t)$ of equation (3.67) should be multiplied by μ_0/μ and the denominator of the integrands of equations (3.66)–(3.69) should be replaced by $\kappa_0 + \mu_0\kappa/\mu$.

Let b_a be the smallest value of b for which the approximation $\kappa_0 \approx b$ is acceptable. According to the discussion in Section 3.2 b_a is very small for frequencies significant in connection with geomagnetic variations. Therefore for $0 \leqslant b \leqslant b_a$ and for reasonable values of the geophysical parameters the following approximations are valid: $e^{-bh} \approx 1$, $e^{-\kappa_0 h} \approx 1$, $b + \kappa \approx \kappa$ ($\approx ik$), $\kappa_0 + \kappa \approx \kappa$ ($\approx ik$). Hence

$$\begin{aligned} \int_0^\infty \frac{se^{-\kappa_0 h}}{\kappa_0 + \kappa} \cos bx db &= \int_0^{b_a} \frac{se^{-\kappa_0 h}}{\kappa_0 + \kappa} \cos bx db + \int_{b_a}^\infty \frac{se^{-\kappa_0 h}}{\kappa_0 + \kappa} \cos bx db \\ &\approx \int_0^{b_a} \frac{s}{\kappa} \cos bx db + \int_{b_a}^\infty \frac{se^{-bh}}{b + \kappa} \cos bx db \\ &\approx \int_0^{b_a} \frac{se^{-bh}}{b+\kappa} \cos bx db + \int_{b_a}^\infty \frac{se^{-bh}}{b+\kappa} \cos bx db = \int_0^\infty \frac{se^{-bh}}{b+\kappa} \cos bx db . \end{aligned} \tag{3.70}$$

Substituting 1 and κ for s the integrals (3.66) and (3.67) can be approximated respectively, and if s equals b and cos bx is replaced by sin bx, an approximation for equation (3.68) is obtained. Hence approximately

$$E_{My}(x,t) = -\frac{i\omega\mu_0 J e^{i\omega t}}{\pi} \int_0^\infty \frac{e^{-bh}}{b+\kappa} \cos bx db , \tag{3.71}$$

$$B_{Mx}(x,t) = \frac{\mu_0 J e^{i\omega t}}{\pi} \int_0^\infty \frac{\kappa e^{-bh}}{b+\kappa} \cos bx db \tag{3.72}$$

and

$$B_{Mz}(x,t) = -\frac{\mu_0 J e^{i\omega t}}{\pi} \int_0^\infty \frac{b e^{-bh}}{b+\kappa} \sin bx db \,. \tag{3.73}$$

(Equation (3.69) could also be approximated correspondingly.) So the formulae where κ_0 has been replaced by b in equations (3.66)–(3.68) have been »derived» or rather made believable in a mathematically very inexact way.

LAW and FANNIN, 1961, discuss the same approximation of E_{My} with their own notations in an appendix. According to their result a typical error in a practical case could be 0.1 %. PARK, 1973, also makes the corresponding approximattion referring to Law's and Fannin's, paper. Law's and Fannin's evaluation, however, contains three errors: 1) The last term in their formula (20) should have A^2 instead of jA. 2) The last term in formula (21) should have A^2 instead of A. 3) Even if formulas (20) and (21) were correct, the calculations of the error at the end of their appendix gives a result which is four times too large. By correcting the first two errors, we can see that the calculation should yield equal expressions for both the true and the approximate integral. To obtain a difference an additional term proportional to $\log A$ would have to be included in the expression for the true integral. If the calculation is then made correctly, the error of the approximation is found to diminish at least to one tenth of that given by Law and Fannin.

On the other hand, this error, as well as Law's and Fannin's »incorrect error», concerns only the part of the integral where the approximation is greatest and not the whole integral. In order to get reliable information on the validity of an approximation the final approximated result should always be compared to the original exact expression. Approximating each term in a sum separately or making approximations in several steps may lead to incorrect results. Direct comparison between equations (3.66) and (3.71) is possible by means of the discussion presented in Law's and Fannin's appendix and formula (3.74) below. The error depends on many parameters and is thus quite complicated for general evaluation, but for example with $\omega = 1 s^{-1}$ the comparison indicates typical values less than the order of 0.01 % which are really negligible.

Expressing $1/(b+\kappa)$ as $(1/k^2)(b-\sqrt{b^2-k^2})$ and using formula (A.81) equation (3.71) can be written as

$$E_{My}(x, t) = \frac{i\omega\mu_0 J e^{i\omega t}}{4k}\left[\frac{Y_1(k(x+ih)) - H_1(k(x+ih))}{x+ih} + \frac{Y_1(k(-x+ih)) - H_1(k(-x+ih))}{-x+ih} + \frac{4(x^2-h^2)}{\pi k(x^2+h^2)^2}\right] \tag{3.74}$$

where Y_1 is the Neumann function of order one and H_1 is the Struve function of order one (see Section A.6). It follows from equations (3.71)–(3.73) that

$$B_{Mx}(x, t) = -\frac{1}{i\omega}\frac{\partial E_{My}(x, t)}{\partial h} + \frac{\mu_0 J e^{i\omega t}}{\pi}\frac{h}{h^2+x^2} \tag{3.75}$$

and

$$B_{Mz}(x, t) = -\frac{1}{i\omega}\frac{\partial E_{My}(x, t)}{\partial x}\,. \tag{3.76}$$

Formula (3.76) is satisfied by the exact expressions (3.60) and (3.62), whose special cases are equations (3.66) and (3.68). Law and Fannin, 1961, and Park, 1972 and 1973, also express the field in terms of the Struve and Neumann functions.

Let us now return to the exact formulae (3.66)–(3.68) for the electromagnetic field on the earth's surface. Then

$$\lim_{\sigma\to\infty} E_{My}(x, t) = 0\,, \tag{3.77}$$

$$\lim_{\sigma\to\infty} B_{Mx}(x, t) = \frac{\mu_0 J e^{i\omega t}}{\pi}\int_0^\infty e^{-\kappa_0 h}\cos bx\,db = -\frac{i\mu_0 k_0 h J e^{i\omega t}}{2\sqrt{x^2+h^2}} H_1^{(2)}(k_0\sqrt{x^2+h^2}) \tag{3.78}$$

and

$$\lim_{\sigma\to\infty} B_{Mz}(x, t) = 0\,. \tag{3.79}$$

The limit $\lim_{\sigma\to\infty}\kappa = \infty$ has been utilized here, and equation (A.76) has been employed in the derivation of equation (3.78). Equations (3.77)–(3.79) could be directly determined using the principle of images. It states that the electromagnetic field outside (or at) a perfectly conducting surface due to a source outside the surface can be calculated simply by imagining an opposite source on the other side of the surface and removing the surface(see *e.g.* FEYNMAN *et al.*, 1964, pp. 6–8–6–11, STRATTON, 1941, pp. 193–194 and pp. 582–583, UMAN, 1969, pp. 48–61). In the case in question the image source is an opposite line current $\bar{j}_{im} = -Je^{i\omega t}\,\delta(z-h)\,\delta(x)\hat{e}_y$ at depth h below the earth's surface (*cf.* KELLER and FRISCHKNECHT, 1970, p. 300). It follows from equation (3.78) that

$$\lim_{\omega\to 0}\,(\lim_{\sigma\to\infty} B_{Mx}(x,t) = \frac{\mu_0 hJ}{\pi(x^2+h^2)} \tag{3.80}$$

and according to equations (3.77) and (3.79) the double-limits $\lim_{\omega\to 0}\,(\lim_{\sigma\to\infty} E_{My}(x,t))$ and $\lim_{\omega\to 0}\,(\lim_{\sigma\to\infty} B_{Mz}(x,t))$ are zero.

Let us take the limit processes in equations (3.66)–(3.68) in reverse order: first

$$\lim_{\omega\to 0} E_{My}(x,t) = 0\,, \tag{3.81}$$

$$\lim_{\omega\to 0} B_{Mx}(x,t) = \frac{\mu_0 J}{2\pi}\int_0^\infty e^{-bh}\cos bx\,db = \frac{\mu_0 hJ}{2\pi(x^2+h^2)} \tag{3.82}$$

and

$$\lim_{\omega\to 0} B_{Mz}(x,t) = -\frac{\mu_0 J}{2\pi}\int_0^\infty e^{-bh}\sin bx\,db = -\frac{\mu_0 xJ}{2\pi(x^2+h^2)} \tag{3.83}$$

Since the right-hand sides of formulae (3.81)–(3.83) do not depend on σ, they also represent the double-limits $\lim_{\sigma\to\infty}(\lim_{\omega\to 0})$. The right-hand sides of equations (3.81)–(3.83) are equal to the value of the primary field in the limit where ω goes to zero (see equations (3.7) and (3.8)).

So the value of the magnetic field in the limit where the conductivity of the earth approaches infinity and the angular frequency approaches zero depends on the order of the limit processes. Similar situations were also mentioned in Section 2.6. This »peculiar» behaviour seems to show that the solution of the problem of a horizontal line direct current above a perfectly conducting half-space is not unambiguous. The results could be interpreted as follows: if the conductivity of the earth approaches infinity, the current induced within the earth shrinks to a surface current. In the limit where the angular frequency approaches zero this surface current becomes a non-zero direct current thus affecting the magnetic field. If, on the other hand, ω first approaches zero, the primary electric field and the current in the earth vanish and the earth has no influence on the electromagnetic field. The second limit, where σ approaches infinity, does not help the situation. The analogous phenomenon could be revealed in the case of a plane wave primary field, too.

The assumptions of finite conductivity of the earth and of positive angular frequency at the beginning of this section, of course, do not prevent the limits $\sigma \to \infty$ and $\omega \to 0$ being taken. Let us point out that the limit processes in formulae (3.77)–(3.79) and (3.81)–(3.83) are simply performed by taking the corresponding limits of the integrands of equations (3.66)–(3.68). The justification of the use of such a method is not proved mathematically here. The only proof is that the results obtained are, according to the above interpretation, physically reasonable.

3.4 Induction in an earth having arbitrarily changing properties in the vertical direction

When dealing with the induction associated with a plane wave primary field it was seen in Section 2.7 that the relationship between the magnetic field and the electric field on the earth's surface is expressible in terms of a surface impedance for every vertical variation of the electromagnetic properties of the earth. The surface impedance only depends on the structure of the earth and, of course, changes as the dependence of the electromagnetic parameters σ, ϵ and μ on the z-coordinate changes. This suggests that formulae (3.60)–(3.64) might also be formally extended to any vertical variation of the properties of the earth. As in Section 2.7 let us, however, assume that the conductivity is everywhere finite. As above, the primary electromagnetic field is expressed by equations (3.5) and (3.6) and the secondary field in the air by formulae (3.23), (3.25) and (3.26). Both involve the assumption of a (slight) conductivity of the air.

The electromagnetic field inside the earth must be calculated by utilizing the more general Maxwell equations (2.93)–(2.96). Taking into account the fact that no y-dependence occurs, equation (2.98), which is a consequence of formulae (2.93), (2.94) and (2.96), yields for the y-component of the electric field

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} - \frac{1}{\mu}\frac{d\mu}{dz}\frac{\partial E_y}{\partial z} + k^2 E_y = 0 . \qquad (3.84)$$

This equation can be solved analogously with the treatment of equation (3.9), *i.e.* using the separation of variables. The x-dependent differential equation will be exactly the same as formula (3.12), and so is its treatment. The z-dependent equation, however, now has the form

$$\frac{d^2 g}{dz^2} - \frac{1}{\mu}\frac{d\mu}{dz}\frac{dg}{dz} + (k^2 - b^2)g = 0 . \qquad (3.85)$$

Equation (3.85) is a linear and homogeneous differential equation of the second order with respect to z. All its solutions are linear combinations of two linearly independent solutions, denoted (for example) by $g_b^+(z)$ and $g_b^-(z)$. As in Section 2.7, the dependence of μ and $\sigma + i\omega\epsilon$ on z should be known precisely in order that the functions $g_b^+(z)$ and $g_b^-(z)$ could be studied thoroughly, and so all discussion of the directions of attenuation and of phase and energy propagations with respect to z is omitted here. (The comments made in Section 3.2 concerning the x-dependence are, of course, valid.)

The general solution for E_y is again clearly obtained by integrating over b (MORSE and FESHBACH, 1953, pp. 497–498). Equation (3.85) shows that $g_b^\pm(z)$ can be regarded as equal to $g_{-b}^\pm(z)$ for negative values of b, and so the extension is, as in Section 3.2, possible for negative values in the integration. Hence

$$E_y(x,z,t) = e^{i\omega t} \int_{-\infty}^{\infty} (D(b)g_b^-(z) + G(b)g_b^+(z))e^{ibx}db\,. \tag{3.86}$$

Equations (2.93) and (3.86) yield that

$$H_x(x,z,t) = \frac{e^{i\omega t}}{i\omega\mu} \int_{-\infty}^{\infty} \left(D(b)\,\frac{dg_b^-(z)}{dz} + G(b)\,\frac{dg_b^+(z)}{dz}\right) e^{ibx}db \tag{3.87}$$

and

$$H_z(x,z,t) = -\frac{e^{i\omega t}}{\omega\mu} \int_{-\infty}^{\infty} b(D(b)g_b^-(z) + G(b)g_b^+(z))e^{ibx}db\,. \tag{3.88}$$

Maxwell's equations (2.93)–(2.96), with the assumptions made, do not couple the components E_x, E_z and H_y to E_y, H_x and H_z, and, as in Section 3.2, the former are considered to be zero. The validity of equation (2.93), which causes formula (2.95) to be satisfied, was included in the above calculation of H_x and H_z. Equation (2.96) is trivially satisfied by (E_y, H_x, H_z), the right-hand side being equal to zero. The total volume charge density in the earth is thus zero, and according to formula (2.97) so is all kind of volume charge. No charge can exist on the planes of discontinuity, including the earth's surface, either, because $\bar{E}$ and $\bar{D}$ are parallel to the xy-plane (see Section B.7). Finally the validity of formula (2.94) is a consequence of equations (2.93), (2.96) and (2.98).

Let us assume that the earth consists of n horizontal regions in each of which the properties of the earth vary continuously with respect to z. The solution (3.86)–(3.88) has to be obtained separately for each layer. The functions $g_b^+(z)$ and $g_b^-(z)$ are different in different regions, which could be recognized by denoting $g_{jb}^+(z)$ and $g_{jb}^-(z)$ (j = 1,..., n). It again seems reasonable to accept in the undermost layer only one fixed linear combination of the possible z-dependencies, say $g_b^+(z)$. The solutions for different regions are coupled by boundary conditions, whose total number is 2n−2. The boundary conditions are treated in exactly the same way as formulae (3.37)–(3.40) in Section 3.2. So for each b 2n−2 equations involving 2n−1 unknown ($D_1(b)$, $G_1(b)$, $D_2(b)$,..., $G_{n-1}(b)$, $G_n(b)$) are obtained. They yield the solutions

$$D_j(b) = \gamma_j(b)\, G_1(b) \tag{3.89}$$

and

$$G_j(b) = \xi_j(b) G_1(b) \tag{3.90}$$

(j = 1,..., n) analogously to formulas (3.45) and (3.46). If E_y and H_x at the earth's surface on its lower side are expressed as equations (3.47) and (3.48), $E_y(b,x,t)$ and $H_x(b,x,t)$ have to be defined as

$$E_y(b,x,t) = G_1(b)(\gamma_1(b)g_b^- + g_b^+)e^{ibx}e^{i\omega t} \tag{3.91}$$

and

$$H_x(b,x,t) = \frac{G_1(b)}{i\omega\mu_1}(\gamma_1(b)g_b^{-\prime} + g_b^{+\prime})e^{ibx}e^{i\omega t} \tag{3.92}$$

where g_b^+, g_b^-, $g_b^{+\prime}$ and $g_b^{-\prime}$ are the values of the functions $g_b^+(z)$ and $g_b^-(z)$ connected with the uppemost layer of the earth and of their derivatives at $z = 0$ and μ_1 is the value of the permeability $\mu(z)$ of the earth at $z = 0$. The surface impedance is now

$$Z(b) = -\frac{E_y(b,x,t)}{H_x(b,x,t)} = -i\omega\mu_1 \frac{\gamma_1(b)g_b^- + g_b^+}{\gamma_1(b)g_b^{-\prime} + g_b^{+\prime}} \tag{3.93}$$

It is only a function of the properties of the earth and is obtained assuming a harmonic time-dependence and no y-dependence. Since the functions $g_b^+(z)$ and $g_b^-(z)$ are even with respect to b, the quantities $\gamma_j(b)$ and $\xi_j(b)$, especially $\gamma_1(b)$, have the same property. This further implies that equation (3.52) is also satisfied by the present surface impedance (3.93).

The boundary conditions at the earth's surface are the same as equations (3.35) and (3.36) if their right-hand sides are replaced by formulae (3.86) and (3.87), respectively, with $z = 0$ and omitting the factor $e^{i\omega t}$. The boundary conditions yield two equations identical to formulae (3.54) and (3.55) with $-(\gamma_1(b)g_b^- +$

$g_b^+)G_1(b)$ and $(-1/\mu_1)(\gamma_1(b)g_b^{-\prime} + g_b^{+\prime})G_1(b)$ substituted for $-(1 + \alpha_1(b))G_1(b)$ and $(\kappa_1/\mu_1)(1-\alpha_1(b))G_1(b)$, respectively. The equations so obtained combined with formula (3.93) give the solution (3.56) exactly. This proves that equations (3.57)–(3.59) are formally correct for any vertical variation of the electromagnetic parameters of the earth, and the validity of formulae (3.60)–(3.64) can then be shown as in the discussion in Section 3.2. So the »guess» made at the beginning of this section was correct. Notice that the discussion about the conductivity of the air after equations (3.60)–(3.62) is also true in the present more general case.

Let us still consider the ratios of the integrands of formulas (3.57)–(3.59) associated with a fixed value of b, *i.e.* the ratios of the Fourier components of E_{My}, B_{Mx} and B_{Mz}. They are

$$\frac{E_{My}(b,x,t)}{B_{Mx}(b,x,t)} = -\frac{Z(b)}{\mu_0}, \tag{3.94}$$

$$\frac{E_{My}(b,x,t)}{B_{Mz}(b,x,t)} = -\frac{\omega}{b} \tag{3.95}$$

and

$$\frac{B_{Mx}(b,x,t)}{B_{Mz}(b,x,t)} = -\frac{\omega\mu_0}{bZ(b)}. \tag{3.96}$$

Since $Z(b)$ only depends on the properties of the earth and on the assumptions of harmonic time-dependence and of no y-dependence, the ratios (3.94)–(3.96) are independent of the properties of the air and of the primary source, if the permeability of the air at the earth's surface is denoted by μ_0. The last requirement is not needed for equation (3.95), which, in fact, is valid everywhere, not only on the earth's surface (see formulae (3.24) and (A.9)).

An impedance, whose dimension is »ohm», is always obtained when an electric field component is divided by a (non-zero) magnetic field intensity component. So the concept of impedances is acceptable for any space- and time-dependencies of the fields and not only in the cases discussed.

The electromagnetic field caused by an infinitely long line current oscillating harmonically with time was treated above assuming that the electromagnetic parameters of the earth, described by the lower half-space, are arbitrary functions of depth. In Section 4.7 one more possible generalization is included.

4. *Induction in the case of a line current primary source oscillating harmonically in time and space*

4.1 Description of the model

As in Chapters 2 and 3, we describe the earth as the lower half-space and the earth's surface as an infinite plane. The upper half-space, the air, is originally assumed to behave electromagnetically as free space, but later in this chapter the air will again be provided with some conductivity. The primary source of the electromagnetic field is assumed to be a similar infinitely long (true) straight line current as in Chapter 3, which is thus situated parallel to and at some height above the earth's surface in the air and which oscillates harmonically with time, but in addition it now has a harmonic space-dependence along the line as well. Let us also at first allow the possibility of exponential attenuation and growth along the line, which, however, has to be rejected later. The space-dependence in the direction of the current necessarily implies the existence of primary (true) charge (see equation of continuity (B.18)).

The same Cartesian coordinate system as in Chapters 2 and 3 is again used, *i.e.* the x- and y-axes point northward and eastward, respectively, the z-axis points downward, and the earth's surface is the plane $z = 0$. Let us assume that the primary current flows in the direction of the y-axis in the plane $x = 0$, so that the expression of the current density is

$$\bar{j} = Je^{i(\omega t - qy)}\delta(z + h)\delta(x)\hat{e}_y \ , \tag{4.1}$$

where J is a complex constant yielding the magnitude and the phase of the current, the δ's are delta functions (equations (A.3) and (A.4)), $h\,(>0)$ is the height of the current from the earth's surface and $\omega\,(>0)$ is the angular frequency. The constant parameter q implying the space dependence along the current can be called a longitudinal propagation constant. As mentioned, exponential attenuation and growth in the y-direction is also possible, and thus q is complex. If both the longitudinal phase propagation and the longitudinal damping take place in the positive y-direction, the following condition must be satisfied:

$$-\frac{\pi}{2} \leqslant \arg q \leqslant 0 \ . \tag{4.2}$$

In fact for $\arg q = -\pi/2$ and $\arg q = 0$ there is no propagation or attenuation, respectively.

It follows from the equation of continuity (B.18) that a (true) charge density

$$\rho = \frac{q}{\omega} J e^{i(\omega t - qy)} \delta(z + h) \delta(x) \tag{4.3}$$

is associated with the primary source current (4.1). In addition there could be a time-independent primary charge, which would not be coupled to the current, but let us assume that this is not the case (cf. Section 3.1). It was indicated above that a non-vanishing conductivity will later be assigned to the air. This conductivity, denoted by σ_0, changes the form of the equation of continuity (B.18), a consequence of which is that the coefficient q/ω in formula (4.3) is replaced by $q/(\omega - i\sigma_0/\epsilon_0)$, and the possible additional charge, which is, however, assumed to vanish, would be exponentially damping with time (see equation (B.27) and Section 3.2).

If q is not purely real, the primary source current and charge are exponentially infinitely large at $y = -\infty$, which involves an unphysical idealization. This infinity comes in addition to the infinity caused by the delta functions. There is also a transverse discontinuity of the source current and charge (*cf.* Section 3.1).

The line current and charge have an east-west orientation, but as in Chapter 3, a suitable rotation of the coordinate system permits the treatment which follows to be used in the discussion of any direction of the primary source parallel to the earth's surface.

The problem will again be treated in the same way as in Chapters 2 and 3: The primary electromagnetic field will first be calculated, and the unknown constants appearing in the expressions of the secondary field above the earth and of the field within the earth, which are solved from Maxwell's equations, will then be coupled to each other and to the primary field by boundary conditions. Again there seem to be different ways of calculating the primary field. In the next section we shall consider the second way mentioned in Chapter 3 where the primary source current and charge initially have a finite thickness which then goes to zero. In Section 4.3 direct integration is performed over the primary source.

4.2 Calculation of the primary field using the »tube» method

Let us denote the finite thickness of the primary source by a and use a cylindrical coordinate system (ρ, φ, z) with the z-axis along the axis of the current and charge »tube» and $\hat{e}_z$ pointing in the same direction as $\hat{e}_y$ of the original coordinate system. The (true) source current can then be expressed as

$$\bar{j} = \begin{cases} \dfrac{J}{\pi a^2} e^{i(\omega t - qz)} e_z \,, & \rho \leqslant a \\ 0 \,, & \rho > a \,. \end{cases} \tag{4.4}$$

The (true) charge density associated with this current and corresponding to formula (4.3) is now

$$\rho_c = \begin{cases} \dfrac{qJ}{\pi a^2 \omega} \, e^{i(\omega t - qz)} \,, & \rho \leqslant a \\[2ex] 0 \,, & \rho > a \,, \end{cases} \tag{4.5}$$

where the subscript c is used to differentiate ρ_c from the coordinate ρ. (All time-independent charge is again rejected.) The delta functions have disappeared from the expressions of $\bar{j}$ and ρ_c, but the infinity at $z = -\infty$ for $-\pi/2 \leqslant \arg q < 0$ still exists. The idealized abrupt transverse change has been removed from $\rho = 0$ to $\rho = a$. In this section, where the primary electromagnetic field only is discussed, the earth is not taken into account and all space has the electromagnetic parameters $\sigma_0 = 0$, ϵ_0 and μ_0 of free space.

Employing the natural assumption that the field caused by the current (4.4) and charge (4.5) has the same time-dependence $e^{i\omega t}$ as these it is easily and exactly similarly to the derivation of formula (B.43) obtained from Maxwell's equations (B.1), (B.3) and (B.4) the inhomogeneous wave equation

$$\nabla^2 \bar{E} + k_0^2 \bar{E} = i\omega\mu_0 \bar{j} + \frac{1}{\epsilon_0} \nabla \rho_c \,, \tag{4.6}$$

where k_0 is defined by equation (B.91).

The electromagnetic field produced by the »tube» evidently has the z-dependence of the sources $\bar{j}$ and ρ_c, *i.e.* e^{-iqz}. All φ-derivatives are clearly zero. Then using equations (4.4) and (4.5) equation (4.6) yields

$$\frac{\partial^2 E_z}{\partial(\eta\rho)^2} + \frac{1}{\eta\rho} \frac{\partial E_z}{\partial(\eta\rho)} + E_z = \frac{i\omega\mu_0 J}{\pi a^2 k_0^2} \, e^{i(\omega t - qz)} \tag{4.7}$$

for $\rho < a$. (For $\rho > a$ the right-hand side of equation (4.7) is zero. The surface $\rho = a$, where a discontinuity exists and where boundary conditions have to be used, is not included in Maxwell's equations.) The quantity η is defined here by the equation

$$\eta^2 = k_0^2 - q^2 \tag{4.8}$$

and by the assumption

$$-\frac{\pi}{2} < \arg \eta \leqslant \frac{\pi}{2} \,. \tag{4.9}$$

To avoid a special case let us assume that q and k_0 are not equal. This assumption is made throughout this whole chapter. The homogeneous differential equation corresponding to equation (4.7) is Bessel's equation of the zeroth order (equation (A.20)). Its solution, acceptable in the region $\rho < a$, is the Bessel function $J_0(\eta\rho)$ (see Section A.6). The other linearly independent solution, the Neumann function $Y_0(\eta\rho)$, approaches infinity as ρ approaches zero. A special solution to equation (4.7) is $(i\omega\mu_0 J/\pi a^2 k_0^2)e^{i(\omega t-qz)}$ so that the complete solution for $E_z(\rho<a)$ is

$$E_z(\rho, z, t) = \left(CJ_0(\eta\rho) + \frac{i\omega\mu_0 J}{\pi a^2 k_0^2}\right) e^{i(\omega t-qz)} \tag{4.10}$$

where C is an unknown complex integration constant.

In order to find out the other components of the electromagnetic field, let us express Maxwell's equations (B.3) and (B.4) (for $\rho < a$) in component form in cylindrical coordinates utilizing the facts that $\partial/\partial t = i\omega$, $\partial/\partial\varphi = 0$ and $\partial/\partial z = -iq$:

$$iqE_\varphi = -i\omega B_\rho \; , \tag{4.11}$$

$$iqE_\rho + \frac{\partial E_z}{\partial\rho} = i\omega B_\varphi \; , \tag{4.12}$$

$$\frac{1}{\rho}\frac{\partial(\rho E_\varphi)}{\partial\rho} = -i\omega B_z \; , \tag{4.13}$$

$$iqB_\varphi = i\omega\mu_0\epsilon_0 E_\rho \; , \tag{4.14}$$

$$iqB_\rho + \frac{\partial B_z}{\partial\rho} = -i\omega\mu_0\epsilon_0 E_\varphi \tag{4.15}$$

and

$$\frac{1}{\rho}\frac{\partial(\rho B_\varphi)}{\partial\rho} = \frac{\mu_0 J}{\pi a^2} e^{i(\omega t-qz)} + i\omega\mu_0\epsilon_0 E_z \; . \tag{4.16}$$

In the present case Maxwell's equations (B.1) and (B.2) are immediate consequences of equations (B.3) and (B.4), respectively (*cf.* the comment after equations (B.35)–(B.38)). Equations (4.11)–(4.16) constitute two independent systems of differential equations, namely (4.11), (4.13), (4.15) and (4.12), (4.14), (4.16). The former contain the components E_φ, B_ρ and B_z and the later E_ρ, E_z and B_φ. Only the latter equations are coupled to the primary source (through (4.16)). So E_φ, B_ρ and B_z are considered zero. It follows from equations (4.12) and (4.14) that

$$E_\rho = \frac{q}{i\eta^2}\frac{\partial E_z}{\partial \rho} \tag{4.17}$$

and

$$B_\varphi = \frac{\omega\mu_0\epsilon_0}{i\eta^2}\frac{\partial E_z}{\partial \rho} . \tag{4.18}$$

Then, using equation (4.10) we obtain for $\rho < a$

$$E_\rho(\rho, z, t) = -\frac{q}{i\eta} CJ_1(\eta\rho)e^{i(\omega t - qz)} \tag{4.19}$$

and

$$B_\varphi(\rho, z, t) = -\frac{\omega\mu_0\epsilon_0}{i\eta} CJ_1(\eta\rho)e^{i(\omega t - qz)} . \tag{4.20}$$

Formula (A.40) was utilized here.

In order to be accurate and careful it should still be checked that the electromagnetic field obtained utilizing the inhomogeneous wave equation (4.6) really satisfies all Maxwell's equations (*cf.* the comment after equations (B.43) and (B.44)). The validity of the φ-component of equation (B.3) and the ρ-component of equation (B.4) is involved in the derivation of equations (4.17) and (4.18). The validity of the z-component of equation (B.4) can be seen simply by direct substitution of the expressions (4.10) and (4.20). The other components of equations (B.3) and (B.4) are trivially zero, and because, as mentioned, in the case in question equations (B.1) and (B.2) are direct consequences of the latter two Maxwell equations all Maxwell's equations are thus satisfied.

The calculation of the electromagnetic field outside the »tube» can obviously be carried out in the same way as inside. Now Maxwell's equations (B.1)–(B.4) do not contain any source terms and consequently equations (4.6) and (4.7) are homogeneous. The treatment of components E_φ, B_ρ and B_z would again be independent of that of E_ρ, E_z and B_φ, which ultimately proves that the former are everywhere zero as the field is caused by the source (4.4) and (4.5). All possible coupling occurs through Maxwell's equations or through continuity conditions at surfaces of discontinuity.

An arbitrary solution of the homogeneous equation (4.7) can, as indicated above, be expressed as a linear combination of $J_0(\eta\rho)$ and $Y_0(\eta\rho)$. It is equally possible to express the solution as a linear combination of the Hankel functions of the first and of the second kind and of the zeroth order $H_0^{(1)}(\eta\rho)$ and $H_0^{(2)}(\eta\rho)$ (see equations (A.23) and (A.24)).

Formulae (4.2), (4.8), (4.9) and (B.91) imply that the argument of η satisfies the condition $0 \leqslant \arg\eta \leqslant \pi/2$. If $\arg\eta$ equals zero, which is achieved when q is real and less than k_0 or $\arg q = -\pi/2$, the Hankel function $H_\nu^{(1)}(\eta\rho)$ with the time-dependence $e^{i\omega t}$ represents asymptotically, *i.e.* as ρ goes to infinity, an inward-phase-travelling wave without any exponential attenuation or growth (equation (A.58)). The Hankel function $H_\nu^{(2)}(\eta\rho)$ is connected with a similar outward-travelling wave (equation (A.59)). In fact, both functions approach zero as $\rho^{-1/2}$ as ρ approaches infinity. As the electromagnetic field caused by the z-axis centred current and charge »tube» is discussed, it seems natural to choose the outward-travelling wave, *i.e.* the solution $H_0^{(2)}(\eta\rho)$ of the homogeneous equation (4.7). If $\arg\eta$ is equal to $\pi/2$ corresponding to a real value of q larger than k_0, the asymptotic form of the Hankel function $H_\nu^{(1)}(\eta\rho)$ with the time factor $e^{i\omega t}$ does not include any phase propagation and is attenuated exponentially as ρ increases. Likewise the asymptotic expression of $H_\nu^{(2)}(\eta\rho)$ involves exponential growth without phase propagation as ρ approaches infinity. In this case it seems reasonable to accept the solution $H_0^{(1)}(\eta\rho)$ of the homogeneous equation (4.7).

Let us discuss the general case where the condition $0 < \arg\eta < \pi/2$ is valid, *i.e.* $-\pi/2 < \arg q < 0$. Since the real part of η is positive the functions $H_\nu^{(1)}(\eta\rho)$ and $H_\nu^{(2)}(\eta\rho)$ with the time factor $e^{i\omega t}$ represent asymptotically inward- and outward-phase-travelling waves, respectively. Since the imaginary part of η is also positive, the functions $H_\nu^{(1)}(\eta\rho)$ and $H_\nu^{(2)}(\eta\rho)$ exponentially approach zero and infinity, respectively, as ρ approaches infinity. Thus if infinite growth is avoided, *i.e.* the Hankel function of the second kind is rejected, the field seems to phase propagate in the wrong direction. The total phase propagation is, of course, not purely radial; z-propagation is also present due to the factor e^{-iqz}. The contradiction between the radial phase propagation and attenuation directions indicates that the problem in question has no physically acceptable solution. However, the unphysical exponentially infinite growth of the source current and charge as z approaches $-\infty$ has been assumed. Hence other infinite unphysicalities might also be expected as an effect of this.

All the conclusions above about the behaviour of the solutions involving Hankel functions have been obtained simply by replacing the Hankel functions with the corresponding asymptotic expressions. However, it is known only that the difference between a function and its asymptotic expression is small compared to the absolute value of the asymptotic expression (see Section A.5). Therefore, for example fhe function $H_\nu^{(1)}(\eta\rho) + H_\nu^{(2)}(\eta\rho)$ $(= 2J_\nu(\eta\rho))$ has the same asymptotic representation $(\pi\eta\rho/2)^{-1/2}e^{-i(\eta\rho-\nu\pi/2-\pi/4)}$ as $H_\nu^{(2)}(\eta\rho)$ in the case $0 < \arg\eta < \pi/2$. Hence, in principle, $H_\nu^{(2)}(\eta\rho)$ could asymptotically also involve an inward-travelling wave with a small amplitude compared to the exponentially large quantity $|(\pi\eta\rho/2)^{-1/2}e^{-i(\eta\rho-\nu\pi/2-\pi/4)}|$ without changing formula (A.59). This is a question of the non-uniqueness of asymptotic expressions mentioned in Section A.5.

Further, the approximation of the real, *i.e.* the physical, part of a complex function by the real part of the asymptotic expression of the function in question is not straightforward, since the error included is only small compared with the absolute value of the complex asymptotic expression (equation (A.18)). In the case of a large imaginary part such an approximation could be very wrong. However, we are dealing with complex quantities that are products of one of the above-mentioned Hankel functions and of a ρ-independent part. For such quantities it can be established that the real parts with large values of ρ are close to expressions which are obtained from the asymptotic formulae (A.58) and (A.59), and which oscillate sinusoidally with ρ, as compared to the damping or increasing amplitude of the oscillation.

A similar contradiction between the directions of phase propagation and attenuation can easily be demonstrated in the situation where the field oscillating harmonically with time depends on two Cartesian coordinates. In such a case, which can be considered a limit of the cylindrical situation as the radius of the cylindrical surface approaches infinity, no asymptotic expressions have to be used.

As pointed out, the direction of propagation discussed here is that of constant phase. However, the direction of energy flow, *i.e.* the direction of the Poynting vector, seems more significant (see Section B.9). To examine this direction let us first write the possible expressions for the field outside the »tube» explicitly. With the use of formulae (4.17) and (4.18) acceptance of the Hankel function of the first kind leads to the following expressions:

$$E_\rho(\rho, z, t) = -\frac{q}{i\eta} FH_1^{(1)}(\eta\rho)e^{i(\omega t-qz)}, \tag{4.21}$$

$$E_z(\rho, z, t) = FH_0^{(1)}(\eta\rho)e^{i(\omega t-qz)} \tag{4.22}$$

and

$$B_\varphi(\rho, z, t) = -\frac{\omega\mu_0\epsilon_0}{i\eta} FH_1^{(1)}(\eta\rho)e^{i(\omega t-qz)} \tag{4.23}$$

where F is an unknown complex integration constant. Similarly with the use of the Hankel function of the second kind:

$$E_\rho(\rho, z, t) = -\frac{q}{i\eta} GH_1^{(2)}(\eta\rho)e^{i(\omega t - qz)}\,, \tag{4.24}$$

$$E_z(\rho, z, t) = GH_0^{(2)}(\eta\rho)e^{i(\omega t - qz)} \tag{4.25}$$

and

$$B_\varphi(\rho, z, t) = -\frac{\omega\mu_0\epsilon_0}{i\eta} GH_1^{(2)}(\eta\rho)e^{i(\omega t - qz)} \tag{4.26}$$

where G is an integration constant. It is evident that both equations (4.21)–(4.23) and (4.24)–(4.26) satisfy the proper Maxwell equations, as do all their linear combinations, which involve every possible solution for the field outside the »tube».

It has already been indicated above that if infinite growth for large values of ρ is to be avoided in the case $0 < \arg\eta \leqslant \pi/2$, the only acceptable solution is given by formulae (4.21)–(4.23). Referring to Section C.1 and taking into account that the medium is non-conducting, *i.e.* k_0^2 is real, it is seen that equations (4.21)–(4.23) express a field whose energy flow on average takes place asymptotically in the direction of phase propagation, which with $0 \leqslant \arg\eta < \pi/2$ is inwards in the radial direction. So the discussion of the Poynting vector does not help in the contradiction included in the case $0 < \arg\eta < \pi/2$.

According to what was said above, equations (4.24)–(4.26) seem to give an acceptable solution for $\arg\eta = 0$, in which case no asymptotic exponential growth or attenuation of the Hankel functions occurs. This conclusion is likewise correct when the Poynting vector is investigated, for according to Section C.1 the energy of the electromagnetic field expressed by formulae (4.24)–(4.26) also on average flows asymptotically (in a non-conducting medium) in the direction of phase propagation and hence outwards in the radial direction.

In the case $\arg\eta = \pi/2$ neither the solution (4.21)–(4.23) nor the solution (4.24)–(4.26) involves radial phase propagation asymptotically, and hence, again referring to Section C.1, no radial energy flow takes place. Thus, as indicated above, the solution (4.21)–(4.23), which exponentially approaches zero as ρ approaches infinity, seems acceptable.

Owing to the condition (4.2) the energy flows on average axially in the positive z-direction, *i.e.* in the direction of attenuation and phase propagation, even if the medium were conducting (see Section C.1). Of course, assumptions $Re q = 0$ or $Im q = 0$ involve special cases in which phase propagation, attenuation or energy flow vanish.

The asymptotic behaviour of the energy flow has been studied above referring to Section C.1 in which the asymptotic formulae of the Hankel functions are substituted in the expression of the complex Poynting vector and then the real part is taken. This procedure should be considered mathematically more accurately. We omit such a discussion, however, and regard the procedure as acceptable and correct on the grounds that analogous results can be obtained in the case of a similar electromagnetic field depending on two Cartesian coordinates. In the latter case no approximations are needed.

As a summary of the above discussion it can be said that the electromagnetic field outside the »tube» is obtained acceptably only in two special cases: for $\arg\eta=0$ it is given by formulae (4.24)–(4.26) and for $\arg\eta = \pi/2$ by formulae (4.21)–(4.23). In the general case $0 < \arg\eta < \pi/2$ problems seem to arise. The difficulty is certainly not caused by the »tube» method, but by the assumptions. In Section 4.3 we will show that no finite expression for the primary field is obtained by direct integration either, if the medium, *i.e.* the air, is non-conducting and q has a negative imaginary part. Thus the case $\arg q = -\pi/2$, which was considered acceptable above, has to be excluded. When »suitable» conductivity is assigned to the air, a negative imaginary part of q may be allowed in Section 4.3, and there can be seen no doubt that the »tube» method would also succeed better with the assumption of conducting air. Such treatment is, however, omitted here.

Let us finally study what the primary field would be if the calculation were performed without taking care of the difficulty mentioned, first with formulae (4.21)–(4.23) and then with formulae (4.24)–(4.26). As stated above these are only two possibilities, since the field outside the »tube» might in principle be any linear combination of these solutions. If the outside field is expressed by equations (4.21)–(4.23) the unknown coefficients C and F are obtained from the continuity conditions of $E_z(\rho, z, t)$ and $H_\varphi(\rho, z, t)$ at $\rho = a$, which using formulae (4.10) and (4.20) and dividing by the common factor $e^{i(\omega t - qz)}$ are expressed as

$$CJ_0(\eta a) + \frac{i\omega\mu_0 J}{\pi a^2 k_0^2} = FH_0^{(1)}(\eta a) \tag{4.27}$$

and

$$-\frac{\omega\epsilon_0}{i\eta}\, CJ_1(\eta a) = -\frac{\omega\epsilon_0}{i\eta}\, FH_1^{(1)}(\eta a)\,. \tag{4.28}$$

A question may arise, whether it is sufficient to demand only the continuity of the tangential components of $\bar{E}$ and $\bar{H}$, since the medium does not actually change at $\rho = a$. There is free space (or air) both inside and outside the »tube», and therefore it might be thought that the total fields must be continuous. However, the discussion of Section B.7, where the electromagnetic boundary conditions are derived, is valid for any surface of discontinuity, for example for the present surface $\rho = a$, at which an abrupt change in the current and charge densities occurs (equations (4.4) and (4.5)). Hence the mere tangential continuity of $\bar{E}$ and $\bar{H}$ is actually required (formulae (B.57) and (B.59)). Further it can be seen from formulae (4.19), (4.20), (4.21) and (4.23) that the continuity of the normal component E_ρ is a consequence of the continuity of H_φ.

Equations (4.27) and (4.28) yield

$$F = \frac{\omega\mu_0\eta J J_1(\eta a)}{2k_0^2 a} \tag{4.29}$$

where formula (A.45) was utilized. The expression of C is otherwise similar to that of F, but $J_1(\eta a)$ is replaced by $H_1^{(1)}(\eta a)$. Equations (4.21)–(4.23) and (4.29) thus express the electromagnetic field outside the current »tube» as a function of the radius a of the tube. The desired result, *i.e.* the field caused by a line current situated at the z-axis, is obtained by letting a go to zero, and at this limit

$$E_\rho = \frac{i\omega\mu_0 q\eta J}{4k_0^2} H_1^{(1)}(\eta\rho)e^{i(\omega t - qz)}\ , \tag{4.30}$$

$$E_z = \frac{\omega\mu_0\eta^2 J}{4k_0^2} H_0^{(1)}(\eta\rho)e^{i(\omega t - qz)} \tag{4.31}$$

and

$$B_\varphi = \frac{i\mu_0\eta J}{4} H_1^{(1)}(\eta\rho)e^{i(\omega t - qz)} \tag{4.32}$$

where formula (A.47) with $\nu = 1$ was used. The quantity $\omega\mu_0/k_0^2$ can also be expressed as $1/\omega\epsilon_0$ (see equation (B.91)).

In exactly the same way we obtain from formulae (4.10), (4.20), (4.25) and (4.26) with the boundary conditions at $\rho = a$ that

$$G = -\frac{\omega\mu_0\eta J J_1(\eta a)}{2k_0^2 a} \tag{4.33}$$

where formula (A.46) was utilized. The final result describing the field caused by the line current is now

$$E_\rho = -\frac{i\omega\mu_0 q\eta J}{4k_0^2} H_1^{(2)}(\eta\rho)e^{i(\omega t-qz)}, \tag{4.34}$$

$$E_z = -\frac{\omega\mu_0\eta^2 J}{4k_0^2} H_0^{(2)}(\eta\rho)e^{i(\omega t-qz)} \tag{4.35}$$

and

$$B_\varphi = -\frac{i\mu_0\eta J}{4} H_1^{(2)}(\eta\rho)e^{i(\omega t-qz)}. \tag{4.36}$$

As indicated above, if the condition $0 < \arg\eta < \pi/2$ is satisfied, neither equations (4.30)–(4.32) nor equations (4.34)–(4.36) seem to be acceptable, but equations (4.34)–(4.36) are reasonable in the case $\arg\eta = 0$, and for $\arg\eta = \pi/2$ (4.30)–(4.32) seem to be the correct ones. (As stated, formulae (4.2), (4.8), (4.9) and (B.91) imply that $0 \leqslant \arg\eta \leqslant \pi/2$.) Both sets of equations (4.30)–(4.32) and (4.34)–(4.36) give the same electromagnetic field near the line current, *i.e.* for small values of ρ. This field is expressed by

$$E_\rho = \frac{\omega\mu_0 qJ}{2\pi k_0^2\rho} e^{i(\omega t-qz)} = \frac{qJ}{2\pi\omega\epsilon_0\rho} e^{i(\omega t-qz)}, \tag{4.37}$$

$$E_z = \frac{i\omega\mu_0\eta^2 J}{2\pi k_0^2} \log\eta\rho\, e^{i(\omega t-qz)} = \frac{i\eta^2 J}{2\pi\omega\epsilon_0} \log\eta\rho\, e^{i(\omega t-qz)} \tag{4.38}$$

and

$$B_\varphi = \frac{\mu_0 J}{2\pi\rho} e^{i(\omega t-qz)} \tag{4.39}$$

(see formulae (A.48) and (A.49)).

4.3 Primary field obtained by integration

Let us now leave the »tube» method and consider direct integration over the primary source expressed by equations (4.1) and (4.3), which should, of course, give the same results and conclusions obtained in the previous section. Actually only the primary magnetic field around the source will be calculated by integrating; the electric field (outside the source) is obtained from Maxwell's equation (B.4). The coordinate system introduced in Section 4.2, in which the z-axis lies along the current and charge line, is still used and so formulae (4.1) and (4.3) have to be rewritten as

$$\bar{j} = J e^{i(\omega t - qz)} \delta(x) \delta(y) \hat{e}_z \tag{4.40}$$

and

$$\rho_c = \frac{q}{\omega} J e^{i(\omega t - qz)} \delta(x) \delta(y) \, . \tag{4.41}$$

Actually equation (4.41) is not needed, since charge density does not appear when integrating the magnetic field (see formula (B.81)). All space outside the source behaves electromagnetically as free space, until otherwise assumed.

The use of direct integration in connection with a transverse discontinuity and with an infinity associated with the delta functions is again, as in Section 3.1, accepted referring to the discussions in Sections B.1 and B.8. Another difficulty now arises, because the integral $\int_S da' \hat{n}' \cdot [\bar{j}]/R$ does not vanish when S approaches infinity unless $Im q = 0$. Hence according to Section B.8 formulae (B.78) and (B.79) do not satisfy the Lorentz condition (B.74). The use of formula (B.81) thus seems to be inadmissible. (In fact, as can be shown, it would be sufficient, if both the time derivative and the gradient of the integral vanished, but in the present case the former implies only a multiplication by $i\omega$.) The difficulty is due to the exponential infinity at $z = -\infty$ occurring for $Im q < 0$. Let us therefore assume that the primary current stops (continuously) at a point $z = z_0$ and for values $z \geqslant z_0$ is given by formula (4.40), which, of course, also changes the expression of the charge density. For this current, in which the exponential infinity is avoided, the above integral vanishes and formula (B.81) is applicable. Letting z_0 then approach minus infinity the magnetic field caused by the original line current (4.40) is obviously obtained. In other words, the use of formula (B.81) with equation (4.40) has been »shown correct». The surface S now lies farther in $z = -\infty$ than the end of the line current. Later in this section an assumption of non-zero conductivity of the air will make the integral $\int_S da' \hat{n}' \cdot [\bar{j}]/R$ vanish and then the use of equation (B.81) is better justified.

The substitution of equation (4.40) into formula (B.81) yields

$$\bar{B}(\bar{r}, t) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{Je^{i\omega\left(t - \frac{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}{c}\right)} e^{-iqz'}}{(x-x')^2+(y-y')^2+(z-z')^2} \cdot \tag{4.42}$$

$$\cdot \left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + i\frac{\omega}{c}\right) \delta(x')\delta(y')\hat{e}_z \times ((x-x')\hat{e}_x +$$

$$+ (y-y')\hat{e}_y + (z-z')\hat{e}_z)dx'dy'dz' .$$

(The unit vectors $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$ are the same for both the source point $\bar{r}' = (x', y', z')$ and the observation point $\bar{r} = (x, y, z)$.) By noting that the quantity $\sqrt{x^2+y^2}$ is equal to the radial cylindrical coordinate ρ and the vector $(1/\rho)(-y\hat{e}_x+x\hat{e}_y)$ is the cylindrical unit vector $\hat{e}_\varphi$ and by changing the variable of integration to $u = z'-z$ we get from equation (4.42):

$$\bar{B}(\bar{r}, t) = \frac{\mu_0 J e^{i(\omega t-qz)}}{4\pi} \hat{e}_\varphi \int_{-\infty}^{\infty} \frac{\rho e^{-i\frac{\omega}{c}\sqrt{\rho^2+u^2}}}{\rho^2+u^2} e^{-iqu}\left(\frac{1}{\sqrt{\rho^2+u^2}} + i\frac{\omega}{c}\right) du . \tag{4.43}$$

(Notice that ρ plays the role of r_0 of Section 3.1.) If a function $L(\rho)$ is defined by

$$L(\rho) = \int_{-\infty}^{\infty} \frac{e^{-i\frac{\omega}{c}\sqrt{\rho^2+u^2}}}{\sqrt{\rho^2+u^2}} e^{-iqu}\, du \tag{4.44}$$

equation (4.43) can be written as

$$\bar{B}(\bar{r}, t) = -\frac{\mu_0 J e^{i(\omega t-qz)}}{4\pi} \frac{dL(\rho)}{d\rho} \hat{e}_\varphi . \tag{4.45}$$

At this point we could mention that the conservation of charge requires the existence of charge at the ends of the primary line currents of this chapter as well as of Chapter 3. These charges situated at infinity are not included in the primary charge densities discussed in this paper. Such an omission is permissible in Chapter 3, because the »end charges» do not have any effect on the primary fields. The same is true also now for the charge at $z = +\infty$, but if $Im q < 0$, the »end charge» at $z = -\infty$ is exponentially infinitely large and cannot thus be neglected. However, the charge density does not (explicitly) appear in formula (B.81) and so equations (4.42)–(4.45) are correct. Later when the air is assumed »sufficiently» conducting, the »end charge» at $z = -\infty$ also becomes insignificant.

Consideration of formula (4.44) shows that there is no finite $L(\rho)$ if Imq differs from zero. In the case $Imq = 0, L(\rho)$ evidently has a finite value based qualitatively on the facts that for large values of $|u|$ the integrand of equation (4.44) can be approximated by $e^{-i\frac{\omega}{c}|u|}e^{-iqu}/u$ and that $\int_0^\infty(\sin x/x)dx = \pi/2$. According to the assumption made in the preceding section $\underline{q}$ differs from ω/c. The calculation of the primary electromagnetic field thus seems, as in Section 4.2, to run into difficulties, if Imq is negative. In fact, the case $\arg q = -\pi/2$ was considered manageable in Section 4.2. It was mentioned in Section 4.1 that the exponential attenuation and growth along the primary source will be rejected later in this chapter, *i.e.* Imq is set equal to zero. Hence it would seem most natural to make this assumption now. Let us, however, proceed in another way and still allow Imq to have negative values, but assigning a non-zero conductivity σ_0 to the air. This assumption of the conductivity of the air would in any case be made when considering the secondary field in the air (*cf.* Section 3.2).

Referring to Section B.8 it is clear that formulae (4.42)–(4.45) are formally valid in the case of conducting air, provided ω/c is replaced by the new propagation constant k_0 which takes the conductivity σ_0 into account. Thus formula (4.45) is true with $L(\rho)$ expressed by.

$$L(\rho) = \int_{-\infty}^{\infty} \frac{e^{-ik_0\sqrt{\rho^2+u^2}}}{\sqrt{\rho^2+u^2}}\, e^{-iqu}\,du\,. \tag{4.46}$$

As in Section 3.2, the permittivity and the permeability of the air could also have any values in the discussion which follows, but let us, however, keep them equal to the free space values ϵ_0 and μ_0.

It seems that $L(\rho)$ of equation (4.46) exists as finite if Imk_0 is less than or equal to Imq. For $Imk_0 < Imq$ the possible exponential growth of e^{-iqu} is cancelled by the stronger vanishing of $e^{-ik_0\sqrt{\rho^2+u^2}} \approx e^{+ik_0u}$ as u goes to $-\infty$, and for $Imk_0 = Imq$ the evidence of the existence of $L(\rho)$ is, as above, qualitatively based on the integral $\int_0^\infty(\sin x/x)dx = \pi/2$.

If e^{-iqu} is expressed as $\cos qu - i\sin qu$ (formula (A.29)), equation (4.46) can be written in the form

$$L(\rho) = 2\int_0^\infty \frac{e^{-ik_0\sqrt{\rho^2+u^2}}}{\sqrt{\rho^2+u^2}}\cos qu\,du\,, \tag{4.47}$$

since the integrands involving $\cos qu$ and $\sin qu$ are even and odd, respectively. Employment of formula (A.68) now yields

$$L(\rho) = 2K_0((\rho^2(q^2 - k_0^2))^{1/2}) \tag{4.48}$$

where K_0 denotes a modified Bessel function of the zeroth order. The condition $\mathrm{Re}(ik_0\rho \pm iq\rho) > 0$ must be valid. Because ρ is real and positive (outside the source), Imk_0 is negative owing to the conductivity of the air and Imq is non-positive according to formula (4.2), the condition $Re(ik\rho + iq\rho) > 0$ is always satisfied, and equation (4.48) is obtained when

$$Imk_0 < Imq\,. \tag{4.49}$$

The term $\cos(\nu\,\overline{\arctan}\ q/ik_0)$ (ν=0) was set equal to one when obtaining formula (4.48). This is correct, because owing to the assumption in Section 4.2 that q differs from k_0 and to formulae (4.2) and (B.42) for k_0, $\overline{\arctan}\ q/ik_0$ cannot be infinite.

It was concluded above that the equality of Imk_0 and Imq might also be permitted, and as mentioned in Section A.6 the equality sign could and should obviously be added to formula (4.49). The fact, also indicated in Section A.6, that formula (4.48) is found in literature for $Imk_0 = Imq = 0$ is emphasized here, too. However, since equation (4.48) is not definitely shown to be true for $Imk_0 = Imq < 0$ in this work, let us keep to the stricter condition (4.49), and later, when Imq equals zero and the air is conducting, it will be satisfied automatically. If the excluded case $Imk_0 = Imq$ were involved in the following treatment, some additional complications would appear, but evidently the discussion would not become impossible.

The inequality (4.49), which implies the existence of $L(\rho)$ as finite, also makes the integral $\int_s da' \hat{n}' \cdot [\bar{j}]/R$ with the »retarded» time $t - k_0R/\omega$ vanish (*cf.* above). Thus according to Section B.8 the Lorentz condition expressed now as formula (B.88) is satisfied by equations (B.78) and (B.79) modified for a conducting medium. This justifies the use of formula (B.81) and no end point $z = z_0$ of the source is needed as above. We might think that the calculation of $\bar{B}$ in non-conducting air and with $Imq < 0$, failed, because, as stated, the Lorentz condition (B.74) is not valid in that case, and so the justification of the use of formula (B.81) is not completely definite. In principle, we could then further suppose that a solution of formulae (B.70) and (B.71), which does not satisfy the Lorentz condition, would also give a finite field in the case $\sigma_0 = 0$ and $Imq < 0$. Let us, however, believe in the »proof» including the assumption of an end point $z = z_0$ and justifying the use of formula (B.81) in the free space case, too. Hence let us, consider $L(\rho)$ (and $\bar{B}$) existing as finite only if $Imk_0 \underset{(-)}{<} Imq$.

It is always very important to specify in which complex half-plane a square root lies (see Section A.2). Let us therefore examine the argument of the modified Bessel function of equation (4.48). It is possible to show that $\rho^2(q^2 - k_0^2)$ cannot, owing to the inequality (4.49), be a non-positive real number. So according to the discussion after formula (A.68) the following condition is valid: $-\pi/2 < \arg(\rho^2(q^2 - k_0^2))^{1/2} < \pi/2$. Expressing $(\rho^2(q^2 - k_0^2))^{1/2}$ as $\rho(q^2 - k_0^2)^{1/2}$, $\arg(\rho^2(q^2 - k_0^2))^{1/2}$ is equal to $\arg(q^2 - k_0^2)^{1/2}$, because ρ is real and positive.

For $-\pi/2 < \arg(q^2 - k_0^2)^{1/2} \leqslant 0$ the inequalities $0 < \arg i(q^2 - k_0^2)^{1/2} \leqslant \pi/2$ are satisfied. Since $(i(q^2 - k_0^2)^{1/2})^2 = k_0^2 - q^2$, the quantity $i(q^2 - k_0^2)^{1/2}$ is then equal to η defined by formulae (4.8) and (4.9) also in the case of conducting air. It now results from equations (4.48) and (A.63) that

$$L(\rho) = \pi i H_0^{(1)}(\eta\rho)\,. \tag{4.50}$$

On the other hand, the inequalities $0 < \arg(q^2 - k_0^2)^{1/2} < \pi/2$ imply that the formula $-\pi/2 < \arg(-i(q^2 - k_0^2)^{1/2}) < 0$ is satisfied. Since $(-i(q^2 - k_0^2)^{1/2})^2 = k_0^2 - q^2$, the quantity $-i(q^2 - k_0^2)^{1/2}$ can then be expressed as η. Formulae (4.48) and (A.64) now yield

$$L(\rho) = -\pi i H_0^{(2)}(\eta\rho)\,. \tag{4.51}$$

As a summary, $L(\rho)$ is given by equation (4.50) or by equation (4.51) depending on whether $\arg\eta$ is greater or less than zero. $\mathrm{Arg}\,\eta$ cannot be equal to zero, because it would make $q^2 - k_0^2$ real and non-positive.

By substituting the expressions of $L(\rho)$ (4.50) and (4.51) in equation (4.45) formulae

$$\bar{B}(\bar{r}, t) = \frac{i\mu_0\eta J}{4} H_1^{(1)}(\eta\rho)e^{i(\omega t - qz)}\hat{e}_\varphi \tag{4.52}$$

or

$$\bar{B}(\bar{r}, t) = -\frac{i\mu_0\eta J}{4} H_1^{(2)}(\eta\rho)e^{i(\omega t - qz)}\hat{e}_\varphi \tag{4.53}$$

are obtained, depending on the argument of η, for the magnetic field. Using Maxwell's equation (B.4) outside the source, $\nabla \times \bar{B} = \mu_0(\sigma_0 + i\epsilon_0\omega)\bar{E} = -(k_0^2/i\omega)\bar{E}$, we then get:

$$\bar{E}(\bar{r}, t) = \frac{\omega\mu_0\eta J}{4k_0^2} (iqH_1^{(1)}(\eta\rho)\hat{e}_\rho + \eta H_0^{(1)}(\eta\rho)\hat{e}_z)e^{i(\omega t - qz)} \tag{4.54}$$

or

$$\bar{E}(\bar{r}, t) = -\frac{\omega\mu_0\eta J}{4k_0^2}(iqH_1^{(2)}(\eta\rho)\hat{e}_\rho + \eta H_0^{(2)}(\eta\rho)\hat{e}_z)e^{i(\omega t-qz)} \tag{4.55}$$

where formula (A.38) has been utilized. (The fact that $\bar{E}(\bar{r}, t)$ also has the time dependence $e^{i\omega t}$ has been employed, too.) In the formal sense equations (4.52)–(4.55) give exactly the same electromagnetic field as formulae (4.30)–(4.36), which were regarded as unphysical in the general case. The important point is, however, that in equations (4.52)–(4.55) a conductivity of the air has been assumed such that condition (4.49) is satisfied.

The assumption made in Section 4.2 that q and k_0 are different, as well as formula (4.49), ensure that η is not zero. Let us, however, consider the behaviour of the electromagnetic field as q approaches k_0. In this limit process the argument of η depends on the manner of the approach of q to k_0. Using formulae (A.48) and (A.49) giving the approximations of the Hankel functions for small arguments, both equations (4.52) and (4.54) and equations (4.53) and (4.55) give the magnetic field $(\mu_0/2\pi\rho)Je^{i(\omega t-k_0z)}\hat{e}_\varphi$ and the electric field $(1/2\pi\epsilon_0\rho)(k_0J/(\omega-i\sigma_0/\epsilon_0))\cdot e^{i(\omega t-qz)}\hat{e}_\rho$ in the limit. So the limit expression of the magnetic field is the magnetic field caused by the line current (4.40) with $q = k_0$, but formally treated as a time-independent and longitudinally constant current, and the limit expression of the electric field is the electric field caused by the line charge (4.41) modified by the conductivity of the air as indicated in Section 4.1 and with $q = k_0$, but formally treated as a time-independent and longitudinally constant charge.

Formulae (4.52)–(4.55), (A.58) and (A.59) show that for both $0 < \arg\eta \leqslant \pi/2$ and $-\pi/2 < \arg\eta < 0$ the electromagnetic field approaches zero exponentially as ρ approaches infinity, but the asymptotic radial phase propagation occurs inwards for $0 < \arg\eta < \pi/2$ and outwards for $-\pi/2 < \arg\eta < 0$. Let us now consider the direction of energy flow, and assume first that $0 < \arg\eta \leqslant \pi/2$, in which case the field is given by equations (4.52) and (4.54). According to the discussion of Section C.1 the average asymptotic energy flow has the expression

$$\langle\bar{N}\rangle \approx \alpha_1(\rho, z)((-\beta_1\eta_1 - \beta_2\eta_2)\hat{e}_\rho + (\beta_1 q_1 + \beta_2 q_2)\hat{e}_z) \tag{4.56}$$

where $\alpha_1(\rho, z)$ is a real and positive function proportional to $|J|^2$. β_1, β_2, η_1, η_2, q_1 and q_2 are the real and the imaginary parts of k_0^2, η and q, respectively. Since ω is positive, β_1 is also positive, and the non-zero conductivity of the air makes β_2 negative. The condition $0 < \arg\eta \leqslant \pi/2$ then implies that q_1 and q_2 must be non-zero, *i.e.* $q_1 > 0$ and $q_2 < 0$ (see formulae (4.2) and (4.8)). So the z-component of the asymptotic form of $\langle\bar{N}\rangle$ is positive.

The consideration of the ρ-component of formula (4.56) is, however, much more significant, because the treatment of ρ involved difficulties when the »tube» method was considered above. Since α_1, β_1 and η_2 are positive, η_1 is non-negative and β_2 is negative, the ρ-component of the right hand side of formula (4.56) can be estimated as follows:

$$\alpha_1(-\beta_1\eta_1 - \beta_2\eta_2) \geqslant \frac{\alpha_1}{\eta_2}(\beta_2(\eta_1^2 - \eta_2^2) - 2\beta_1\eta_1\eta_2) = \frac{2\alpha_1}{\eta_2}(k_1q_1 + k_2q_2)(k_1q_2 - k_2q_1) \tag{4.57}$$

where k_1 and k_2, which necessarily differ from zero, denote the real and imaginary parts of k_0, respectively. Because k_1 and q_1 are positive, and k_2 and q_2 negative, the quantity $(2\alpha_1/\eta_2)(k_1q_1 + k_2q_2)$ is positive. Due to the inequality $0 \leqslant \eta_1\eta_2$ the formula $q_1 \geqslant k_1k_2/q_2$ is satisfied and thus the following estimation is valid: $k_1q_2 - k_2q_1 \geqslant k_1q_2(1 - k_2^2/q_2^2)$. The quantity $1 - k_2^2/q_2^2$ is negative because of formula (4.49) and of the negativeness of k_2 and q_2. Since the product k_1q_2 is also negative, $k_1q_2 - k_2q_1$ is positive. Hence formulae (4.56) and (4.57) give a positive ρ-component for the asymptotic expression of $\langle\bar{N}\rangle$.

If the condition $-\pi/2 < \arg\eta < 0$ is satisfied, the electromagnetic field is expressed by equations (4.53) and (4.55). The asymptotic form of the time average of the Poynting vector is again obtained by means of the discussion in Section C.1 and

$$\langle\bar{N}\rangle \approx \alpha_2(\rho, z)((\beta_1\eta_1 + \beta_2\eta_2)\hat{e}_\rho + (\beta_1q_1 + \beta_2q_2)\hat{e}_z) \tag{4.58}$$

where $\alpha_2(\rho, z)$ is real and positive and proportional to $|J|^2$ and the other parameters have the same meanings as above. In this case q_1 and q_2 may be zero and thus the z-component of the asymptotic form of $\langle\bar{N}\rangle$ is only non-negative. Owing to the inequalities $\alpha_2, \beta_1, \eta_1 > 0$ and $\beta_2, \eta_2 < 0$ the ρ-component $\alpha_2(\beta_1\eta_1 + \beta_2\eta_2)$ of the asymptotic expression of $\langle\bar{N}\rangle$ is directly seen to be positive. The results for this case $-\pi/2 < \arg\eta < 0$, where both exponential attenuation and phase propagation occur radially outwards, could also be obtained more directly from Section C.1.

Hence for both ranges of the argument of η the energy seems to flow asymptotically outwards from the source, which was concluded by referring to Section C.1 where the asymptotic expressions of the Hankel functions are substituted in the formula for the complex Poynting vector and then the real part is taken.

If the imaginary part of q is negative, there is a lower limit above which the value of the conductivity σ_0 of the air must be, in order that the necessary condition (4.49) is satisfied. So if the properties of the air are to be arbitrarily close to those of free space, *i.e.* σ_0 is arbitrarily small, it is necessary for *Imq* to be equal to zero. This conclusion is also valid if the equality sign is added to formula (4.49). Hence let us from now on keep the assumption $Imq = 0$. It results from the non-zero conductivity of the air and from the definition of η (formulae (4.8) and (4.9)) that the inequalities $-\pi/2 < \arg\eta < 0$ are satisfied, and so the electromagnetic field is expressed by equations (4.53) and (4.55).

Let us now calculate the limits of the field as σ_0 approaches zero. If $\omega\sqrt{\mu_0\epsilon_0}$ is greater than q, η approaches the real and positive quantity $\sqrt{\omega^2\mu_0\epsilon_0 - q^2}$, which is the correct η of free space. The electromagnetic field in the limit where σ_0 approaches zero is thus expressed by equations (4.53) and (4.55) with k_0 and η corresponding to free space. The continuity of the Hankel functions is utilized here.

If on the other hand $\omega\sqrt{\mu_0\epsilon_0}$ is smaller than q, η approaches the quantity $-i\sqrt{q^2 - \omega^2\mu_0\epsilon_0}$ where the square root is real and positive. This limit is equal to the opposite number of the correct η of free space, *i.e.* equal to the product of the correct η and $e^{-i\pi}$. Hence the electromagnetic field at the limit where σ_0 approaches zero is given by equations (4.53) and (4.55) with k_0 corresponding to free space and η replaced by the opposite of η corresponding to free space. Use of equation (A.35) shows that the field in the limit is expressed by equations (4.52) and (4.54) with k_0 and η corresponding to free space. The continuity of the Hankel functions is again utilized. The special case $\omega\sqrt{\mu_0\epsilon_0} = q$ is neglected here on the basis of the assumption made in Section 4.2.

»The continuity of physics» indicates that the limits derived give the primary field for free space and »ideal air». This conclusion is supported by the fact that the limit field is exactly the same as was obtained by the »tube» method for $\arg\eta = 0$ and for $\arg\eta = \pi/2$ and is given in formulae (4.30)–(4.32) and (4.34)–(4.36). The discussion of the direction of the asymptotic energy flow of the limit field was already included in Section 4.2. The »tube» method also yielded an »acceptable» solution (4.34)–(4.36) for $\arg q = -\pi/2$ and $\sigma_0 = 0$. This result, however, does not appear when direct integration is used, because the inequality $Imq < Imk_0$ would then be valid. So this solution for the primary field has to be excluded.

As mentioned above, formula (4.48) is definitely valid if both Imk_0 and Imq are zero (see also Section A.6). Hence the primary field in free space and with a purely real longitudinal propagation constant can also be obtained by direct integration, and the result is the same as the limit process above gives. Here Section 3.1 can also be referred to, in which the primary field in free space was calculated in the special case of a zero longitudinal propagation constant. The substitution $q = 0$ into formulae (4.53) and (4.55) with $\sigma_0 = 0$, which express the primary field in free space for $q < \omega\sqrt{\mu_0\epsilon_0} = k_0$, yields equations (3.6) and (3.5). Notice the difference in the coordinate systems.

It has been shown that the primary electromagnetic field has a well-defined expression for non-conducting air, *i.e.* for free space, when q is real. However, as indicated earlier in this section and as was the case in Section 3.2, a non-zero conductivity of the air would at any rate be assumed when discussing the secondary field in the air. So let us keep the conductivity σ_0 non-zero but small.

Finally let us express the primary field (equations (4.53) and (4.55)) in the original coordinate system of this chapter mentioned in Section 4.1:

$$\bar{E}(\bar{r}, t) = -\frac{\omega\mu_0\eta J e^{i(\omega t - qy)}}{4k_0^2}\Big[\frac{iqx}{\sqrt{x^2+(z+h)^2}} H_1^{(2)}(\eta\sqrt{x^2+(z+h)^2})\hat{e}_x + \quad (4.59)$$

$$+ \eta H_0^{(2)}(\eta\sqrt{x^2+(z+h)^2})\hat{e}_y + \frac{iq(z+h)}{\sqrt{x^2+(z+h)^2}} H_1^{(2)}(\eta\sqrt{x^2+(z+h)^2})\hat{e}_z\Big]$$

and

$$\bar{B}(\bar{r}, t) = -\frac{i\mu_0\eta J e^{i(\omega t - qy)}}{4\sqrt{x^2+(z+h)^2}} H_1^{(2)}(\eta\sqrt{x^2+(z+h)^2})[(z+h)\hat{e}_x - x\hat{e}_z]\,. \qquad (4.60)$$

4.4 Induction in a horizontally layered earth

As in Sections 2.2 and 3.2 the earth is assumed to be composed of n horizontal layers (Fig. 1) with constant conductivities σ_j ($\neq\infty$), constant permittivities ϵ_j, constant permeabilities μ_j and thicknesses h_j ($h_n=\infty$). As in Chapters 2 and 3 all fields appearing clearly have the time-dependence $e^{i\omega t}$. Since the earth is laterally homogeneous, a reasonable assumption is that all y-dependence is expressed by the term e^{-iqy}.

The secondary electromagnetic field caused by the earth in the air satisfies Maxwell's equations (B.35)–(B.38) with $\sigma=\sigma_0$, $\epsilon=\epsilon_0$ and $\mu=\mu_0$. Thus no charge is connected with the secondary field in the air. The wave equations (B.43) and (B.44) are also satisfied, and then

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + \eta^2 E_y = 0 \qquad (4.61)$$

where η is defined by formulae (4.8) and (4.9) with k_0 including the non-zero conductivity σ_0.

Equation (4.61) can be solved in exactly the same way as equation (3.9), but k_0 is to be replaced by η, and so κ_0 has to be defined by the following formulae:

$$\kappa_0^2 = b^2 - \eta^2 = b^2 + q^2 - k_0^2 \tag{4.62}$$

and

$$-\frac{\pi}{2} < \arg\kappa_0 \leqslant \frac{\pi}{2}. \tag{4.63}$$

As in Section 3.2, the parameter b is real, and integrals with respect to it are Fourier integral representations in x. At first it is non-negative but is later extended to all real values. Because q is also real and a positive conductivity of the air has already been assumed, both the real and the imaginary parts of κ_0 are positive. So the special case $\kappa_0 = 0$ is excluded. Referring to equation (3.23) the solution for E_y can now be written as

$$E_y(x, y, z, t) = e^{i(\omega t - qz)} \int_{-\infty}^{\infty} D_0(b) e^{\kappa_0 z} e^{ibx} db \tag{4.64}$$

where $D_0(b)$ is an unknown »integration constant» function.

In order to determine the other field components let us write Maxwell's equations (B.37) and (B.38) in component form for the air:

$$-iqE_z - \frac{\partial E_y}{\partial z} = -i\omega B_x , \tag{4.65}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -i\omega B_y , \tag{4.66}$$

$$\frac{\partial E_y}{\partial x} + iqE_x = -i\omega B_z , \tag{4.67}$$

$$-iqB_z - \frac{\partial B_y}{\partial z} = \mu_0(\sigma_0 + i\omega\epsilon_0)E_x , \tag{4.68}$$

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0(\sigma_0 + i\omega\epsilon_0)E_y \tag{4.69}$$

and

$$\frac{\partial B_y}{\partial x} + iqB_x = \mu_0(\sigma_0 + i\omega\epsilon_0)E_z . \tag{4.70}$$

The knowledge of the y-dependence e^{-iqy} has been employed in equations (4.65)–(4.70). Equations (4.65) and (4.70) give

$$E_z = -\frac{iq\dfrac{\partial E_y}{\partial z} + i\omega\dfrac{\partial B_y}{\partial x}}{\eta^2} \tag{4.71}$$

and

$$B_x = -\frac{\mu_0(\sigma_0 + i\omega\epsilon_0)\dfrac{\partial E_y}{\partial z} + iq\dfrac{\partial B_y}{\partial x}}{\eta^2} . \tag{4.72}$$

Using equations (4.67) and (4.68) or analogy E_x and B_z can be written as

$$E_x = -\frac{iq\dfrac{\partial E_y}{\partial x} - i\omega\dfrac{\partial B_y}{\partial z}}{\eta^2} \tag{4.73}$$

and

$$B_z = \frac{\mu_0(\sigma_0 + i\omega\epsilon_0)\dfrac{\partial E_y}{\partial x} - iq\dfrac{\partial B_y}{\partial z}}{\eta^2} . \tag{4.74}$$

The division by η is permissible owing to the assumption introduced in Section 4.2 that k_0 and q are different. This assumption is made valid by the reality of q and by the non-zero conductivity σ_0. In fact, the inequalities $-\pi/2 < \arg\eta < 0$ are true.

Equations (4.71)–(4.74) indicate that the component B_y must also be calculated as E_y above. By starting from the wave equation of the magnetic field (B.44) we obtain in the same way as the derivation of formula (4.64)

$$B_y(x, y, z, t) = e^{i(\omega t - qy)} \int_{-\infty}^{\infty} Q_0(b) e^{\kappa_0 z} e^{ibx} db , \tag{4.75}$$

where $Q_0(b)$ is an unknown »integration constant» function. Substitution of equations (4.64) and (4.75) into formulae (4.71)–(4.74) gives

$$E_x(x, y, z, t) = \frac{e^{i(\omega t - qy)}}{\eta^2} [q \int_{-\infty}^{\infty} bD_0(b) e^{\kappa_0 z} e^{ibx} db + i\omega \int_{-\infty}^{\infty} \kappa_0 Q_0(b) e^{\kappa_0 z} e^{ibx} db] , \tag{4.76}$$

$$E_z(x,y,z,t) = \frac{e^{i(\omega t-qy)}}{\eta^2}\left[-iq\int_{-\infty}^{\infty}\kappa_0 D_0(b)e^{\kappa_0 z}e^{ibx}db + \omega\int_{-\infty}^{\infty} bQ_0(b)e^{\kappa_0 z}e^{ibx}db\right], \tag{4.77}$$

$$B_x(x,y,z,t) = \frac{e^{i(\omega t-qy)}}{\eta^2}\left[-\mu_0(\sigma_0+i\omega\epsilon_0)\int_{-\infty}^{\infty}\kappa_0 D_0(b)e^{\kappa_0 z}e^{ibx}db + \right. \tag{4.78}$$

$$\left. + q\int_{-\infty}^{\infty} bQ_0(b)e^{\kappa_0 z}e^{ibx}db\right]$$

and

$$B_z(x,y,z,t) = \frac{e^{i(\omega t-qy)}}{\eta^2}\left[i\mu_0(\sigma_0+i\omega\epsilon_0)\int_{-\infty}^{\infty} bQ_0(b)e^{\kappa_0 z}e^{ibx}db + \right. \tag{4.79}$$

$$\left. - iq\int_{-\infty}^{\infty}\kappa_0 Q_0(b)e^{i\kappa_0 z}e^{ibx}db\right].$$

As in Chapter 3, referring to MORSE and FESHBACH, 1953, pp. 497–498, we believe that all physical solutions for the seconday field in the air are included in equations (4.64) and (4.75)–(4.79). According to Section B.5 the validity of Maxwell's equations (B.35)–(B.38) has to be controlled when the wave equations (B.43) and (B.44) are used: The validity of the x- and z-components of equations (B.37) and (B.38) is evident from the calculation of E_x, E_z, B_x and B_z. The y-components of these equations, *i.e.* formulae (4.66) and (4.69), were not utilized, but their validity is seen from equations (4.71)–(4.74) and observing that E_y and B_y satisfy the wave equation. The satisfaction of equations (B.35) and (B.36) is a consequence of equations (B.37) and (B.38).

The electromagnetic field within the j^{th} layer of the earth satisfies Maxwell's equations (B.35)–(B.38) with $\sigma = \sigma_j$, $\epsilon = \epsilon_j$ and $\mu = \mu_j$ (j = 1,...,n). Analogously to formulae (4.8) and (4.9) let us define a parameter η_j for each layer by the following formulae:

$$\eta_j^2 = k_j^2 - q^2 \tag{4.80}$$

and

$$-\frac{\pi}{2} < \arg\eta_j \leqslant \frac{\pi}{2}. \tag{4.81}$$

The parameter k_j is the propagation constant of the j^{th} layer. By making the reasonable assumption that the conductivity of the earth differs everywhere from zero the quantities κ_j (j = 1,...,n), defined by

$$\kappa_j^2 = b^2 - \eta_j^2 = b^2 + q^2 - k_j^2 \tag{4.82}$$

and

$$-\frac{\pi}{2} < \arg\kappa_j \leqslant \frac{\pi}{2}, \tag{4.83}$$

necessarily have positive real and imaginary parts, which excludes the special cases $\kappa_j = 0$. All factors η_j are then also non-zero and $-\pi/2 < \arg\eta_j < 0$. By referring to the corresponding solution in Section 3.2 and to the above discussion of the secondary field in the air in this section the expressions for the field in the earth can be written as follows:

$$E_x(x,y,z,t) = \frac{e^{i(\omega t - qy)}}{\eta_j^2}\Big[q \int_{-\infty}^{\infty} b(D_j(b)e^{\kappa_j z} + G_j(b)e^{-\kappa_j z})e^{ibx}db + i\omega \int_{-\infty}^{\infty} \kappa_j(Q_j(b)e^{\kappa_j z} - R_j(b)e^{-\kappa_j z})e^{ibx}db\Big], \tag{4.84}$$

$$E_y(x,y,z,t) = e^{i(\omega t - qy)} \int_{-\infty}^{\infty} (D_j(b)e^{\kappa_j z} + G_j(b)e^{-\kappa_j z})e^{ibx}db, \tag{4.85}$$

$$E_z(x,y,z,t) = \frac{e^{i(\omega t - qy)}}{\eta_j^2}\Big[-iq \int_{-\infty}^{\infty} \kappa_j(D_j(b)e^{\kappa_j z} - G_j(b)e^{-\kappa_j z})e^{ibx}db + \omega \int_{-\infty}^{\infty} b(Q_j(b)e^{\kappa_j z} + R_j(b)e^{-\kappa_j z})e^{ibx}db\Big], \tag{4.86}$$

$$B_x(x,y,z,t) = \frac{e^{i(\omega t - qy)}}{\eta_j^2}\Big[-\mu_j(\sigma_j + i\omega\epsilon_j) \int_{-\infty}^{\infty} \kappa_j(D_j(b)e^{\kappa_j z} - G_j(b)e^{-\kappa_j z})e^{ibx}db + q \int_{-\infty}^{\infty} b(Q(b)e^{\kappa_j z} + R_j(b)e^{-\kappa_j z})e^{ibx}db\Big], \tag{4.87}$$

$$B_y(x,y,z,t) = e^{i(\omega t - qy)} \int_{-\infty}^{\infty} (Q_j(b)e^{\kappa_j z} + R_j(b)e^{-\kappa_j z})e^{ibx}db \tag{4.88}$$

and

$$B_z(x,y,z,t) = \frac{e^{i(\omega t - qy)}}{\eta_j^2} [i\mu_j(\sigma_j + i\omega\epsilon_j) \int_{-\infty}^{\infty} b(D_j(b)e^{\kappa_j z} + G_j(b)e^{-\kappa_j z})e^{ibx}db + \\ - iq \int_{-\infty}^{\infty} \kappa_j(Q_j(b)e^{\kappa_j z} - R_j(b)e^{-\kappa_j z})e^{ibx}db] \tag{4.89}$$

for j = 1,..., n − 1, and

$$E_x(x,y,z,t) = \frac{e^{i(\omega t - qy)}}{\eta_n^2} [q \int_{-\infty}^{\infty} bG_n(b)e^{-\kappa_n z}e^{ibx}db - i\omega \int_{-\infty}^{\infty} \kappa_n R_n(b)e^{-\kappa_n z}e^{ibx}db] , \tag{4.90}$$

$$E_y(x,y,z,t) = e^{i(\omega t - qy)} \int_{-\infty}^{\infty} G_n(b)e^{-\kappa_n z}e^{ibx}db , \tag{4.91}$$

$$E_z(x,y,z,t) = \frac{e^{i(\omega t - qy)}}{\eta_n^2} [iq \int_{-\infty}^{\infty} \kappa_n G_n(b)e^{-\kappa_n z}e^{ibx}db + \\ + \omega \int_{-\infty}^{\infty} bR_n(b)e^{-\kappa_n z}e^{ibx}db] , \tag{4.92}$$

$$B_x(x,y,z,t) = \frac{e^{i(\omega t - qy)}}{\eta_n^2} [\mu_n(\sigma_n + i\omega\epsilon_n) \int_{-\infty}^{\infty} \kappa_n G_n(b)e^{-\kappa_n z}e^{ibx}db + \\ + q \int_{-\infty}^{\infty} bR_n(b)e^{-\kappa_n z}e^{ibx}db] , \tag{4.93}$$

$$B_y(x,y,z,t) = e^{i(\omega t - qy)} \int_{-\infty}^{\infty} R_n(b)e^{-\kappa_n z}e^{ibx}db \tag{4.94}$$

and

$$B_z(x,y,z,t) = \frac{e^{i(\omega t - qy)}}{\eta_n^2} [i\mu_n(\sigma_n + i\omega\epsilon_n) \int_{-\infty}^{\infty} bG_n(b)e^{-\kappa_n z}e^{ibx}db + \\ + iq \int_{-\infty}^{\infty} \kappa_n R_n(b)e^{-\kappa_n z}e^{ibx}db] , \tag{4.95}$$

for the undermost layer j = n. The functions $Q_j(b)$ and $R_j(b)$ (j = 1,..., n) are unknown »integration constants». As above, all physical solutions for the field in the earth are obviously included in these equations (MORSE and FESHBACH, 1953, pp. 497–498). The validity of Maxwell's equations (B.35)–(B.38) with formulae (4.84)–(4.95) is also evident. Since the divergence of $\bar{E}$ vanishes, no volume charges appear inside the layers of the earth. In the present case, however, $\bar{E}$ is not parallel to the boundaries of the layers. Thus the existence of surface charge at these surfaces comes into question (see STRATTON, 1941, p. 483).

As we have established, both $Re\,\kappa_j$ and $Im\,\kappa_j$ (j = 0, 1,..., n) are positive. Since b and q are real, the function $e^{i(\omega t-qy+bx)+\kappa_j z}$ has no attenuation in the x- and y-directions. In z both attenuation and phase propagation take place in the negative z-direction. In the same way the function $e^{i(\omega t-qy+bx)-\kappa_j z}$ is not attenuated with respect to x and y and has both attenuation and phase propagation in the positive z-direction. Thus the directions of attenuation and phase propagation are never opposite in the expressions of the secondary field in the air and of the field in the earth. The discussion of the direction of the energy flow is neglected, because no contradiction between the directions of attenuation and phase propagation exists. The treatment of the Poynting vector in Appendix C is not sufficient for the present case (see also Section 3.2).

The tangential components E_x, E_y, $H_x = B_x/\mu$ and $H_y = B_y/\mu$ of the electromagnetic field are continuous across the boundary surfaces $z = 0$, $z = z_1 = h_1$, $z = z_2 = h_1 + h_2, \ldots, z = z_{n-1} = h_1 + h_2 + \ldots + h_{n-1}$ (Section B.7). Hence using formulae (4.59), (4.60), (4.64), (4.75)–(4.79) and (4.84)–(4.95) (excluding the z-components) the boundary conditions can be written as

$$-\frac{i\omega\mu_0\eta qJxH_1^{(2)}(\eta\sqrt{x^2+h^2})}{4k_0^2\sqrt{x^2+h^2}} + \frac{q}{\eta^2}\int_{-\infty}^{\infty} bD_0(b)e^{ibx}db + \tag{4.96}$$

$$+\frac{i\omega}{\eta^2}\int_{-\infty}^{\infty} \kappa_0 Q_0(b)e^{ibx}db = \frac{q}{\eta_1^2}\int_{-\infty}^{\infty} b(D_1(b)+G_1(b))e^{ibx}db +$$

$$+\frac{i\omega}{\eta_1^2}\int_{-\infty}^{\infty} \kappa_1(Q_1(b)-R_1(b))e^{ibx}db\ ,$$

$$-\frac{\omega\mu_0\eta^2 JH_0^{(2)}(\eta\sqrt{x^2+h^2})}{4k_0^2} + \int_{-\infty}^{\infty} D_0(b)e^{ibx}db = \tag{4.97}$$

$$= \int_{-\infty}^{\infty} (D_1(b)+G_1(b))e^{ibx}db\ ,$$

$$-\frac{i\eta JhH_1^{(2)}(\eta\sqrt{x^2+h^2})}{4\sqrt{x^2+h^2}}-\frac{\sigma_0+i\omega\epsilon_0}{\eta^2}\int_{-\infty}^{\infty}\kappa_0 D_0(b)e^{ibx}db+ \tag{4.98}$$

$$+\frac{q}{\mu_0\eta^2}\int_{-\infty}^{\infty}bQ_0(b)e^{ibx}db=-\frac{\sigma_1+i\omega\epsilon_1}{\eta_1^2}\int_{-\infty}^{\infty}\kappa_1(Q_1(b)-G_1(b))e^{ibx}db+$$

$$+\frac{q}{\mu_1\eta_1^2}\int_{-\infty}^{\infty}b(Q_1(b)+R_1(b))e^{ibx}db\ ,$$

and

$$\frac{1}{\mu_0}\int_{-\infty}^{\infty}Q_0(b)e^{ibx}db=\frac{1}{\mu_1}\int_{-\infty}^{\infty}(Q_1(b)+R_1(b))e^{ibx}db \tag{4.99}$$

for $z=0$, and

$$\frac{q}{\eta_j^2}\int_{-\infty}^{\infty}b(D_j(b)e^{\kappa_j z_j}+G_j(b)e^{-\kappa_j z_j})e^{ibx}db+ \tag{4.100}$$

$$+\frac{i\omega}{\eta_j^2}\int_{-\infty}^{\infty}\kappa_j(Q_j(b)e^{\kappa_j z_j}-R_j(b)e^{-\kappa_j z_j})e^{ibx}db=$$

$$=\frac{q}{\eta_{j+1}^2}\int_{-\infty}^{\infty}b(D_{j+1}(b)e^{\kappa_{j+1}z_j}+G_{j+1}(b)e^{-\kappa_{j+1}z_j})e^{ibx}db+$$

$$+\frac{i\omega}{\eta_{j+1}^2}\int_{-\infty}^{\infty}\kappa_{j+1}(Q_{j+1}(b)e^{\kappa_{j+1}z_j}-R_{j+1}(b)e^{-\kappa_{j+1}z_j})e^{ibx}db\ ,$$

$$\int_{-\infty}^{\infty}(D_j(b)e^{\kappa_j z_j}+G_j(b)e^{-\kappa_j z_j})e^{ibx}db=\int_{-\infty}^{\infty}(D_{j+1}(b)e^{\kappa_{j+1}z_j}+G_{j+1}(b)e^{-\kappa_{j+1}z_j})e^{ibx}db\ , \tag{4.101}$$

$$-\frac{\sigma_j+i\omega\epsilon_j}{\eta_j^2}\int_{-\infty}^{\infty}\kappa_j(D_j(b)e^{\kappa_j z_j}-G_j(b)e^{-\kappa_j z_j})e^{ibx}db+ \tag{4.102}$$

$$+\frac{q}{\mu_j\eta_j^2}\int_{-\infty}^{\infty}b(Q_j(b)e^{\kappa_j z_j}+R_j(b)e^{-\kappa_j z_j})e^{ibx}db=$$

$$=-\frac{\sigma_{j+1}+i\omega\epsilon_{j+1}}{\eta_{j+1}^2}\int_{-\infty}^{\infty}\kappa_{j+1}(D_{j+1}(b)e^{\kappa_{j+1}z_j}-G_{j+1}(b)e^{-\kappa_{j+1}z_j})e^{ibx}db+$$

$$+\frac{q}{\mu_{j+1}\eta_{j+1}^2}\int_{-\infty}^{\infty}b\,(Q_{j+1}(b)e^{\kappa_{j+1}z_j}+R_{j+1}(b)e^{-\kappa_{j+1}z_j})e^{ibx}db$$

and

$$\frac{1}{\mu_j}\int_{-\infty}^{\infty}(Q_j(b)e^{\kappa_j z_j}+R_j(b)e^{-\kappa_j z_j})e^{ibx}db= \tag{4.103}$$

$$=\frac{1}{\mu_{j+1}}\int_{-\infty}^{\infty}(Q_{j+1}(b)e^{\kappa_{j+1}z_j}+R_{j+1}(b)e^{-\kappa_{j+1}z_j})e^{ibx}db$$

for $z = z_j$ where $j = 1, \ldots, n-2$, and

$$\frac{q}{\eta_{n-1}^2}\int_{-\infty}^{\infty}b(D_{n-1}(b)e^{\kappa_{n-1}z_{n-1}}+G_{n-1}(b)e^{-\kappa_{n-1}z_{n-1}})e^{ibx}db+ \tag{4.104}$$

$$+\frac{i\omega}{\eta_{n-1}^2}\int_{-\infty}^{\infty}\kappa_{n-1}(Q_{n-1}(b)e^{\kappa_{n-1}z_{n-1}}-R_{n-1}(b)e^{-\kappa_{n-1}z_{n-1}})e^{ibx}db=$$

$$=\frac{q}{\eta_n^2}\int_{-\infty}^{\infty}bG_n(b)e^{-\kappa_n z_{n-1}}e^{ibx}db-\frac{i\omega}{\eta_n^2}\int_{-\infty}^{\infty}\kappa_n R_n(b)e^{-\kappa_n z_{n-1}}e^{ibx}db\,,$$

$$\int_{-\infty}^{\infty}(D_{n-1}(b)e^{\kappa_{n-1}z_{n-1}}+G_{n-1}(b)e^{-\kappa_{n-1}z_{n-1}})e^{ibx}db=\int_{-\infty}^{\infty}G_n(b)e^{-\kappa_n z_{n-1}}e^{ibx}db\,, \tag{4.105}$$

$$-\frac{\sigma_{n-1}+i\omega\epsilon_{n-1}}{\eta_{n-1}^2}\int_{-\infty}^{\infty}\kappa_{n-1}(D_{n-1}(b)e^{\kappa_{n-1}z_{n-1}}-G_{n-1}(b)e^{-\kappa_{n-1}z_{n-1}})e^{ibx}db+ \tag{4.106}$$

$$+\frac{q}{\mu_{n-1}\eta_{n-1}^2}\int_{-\infty}^{\infty}b(Q_{n-1}(b)e^{\kappa_{n-1}z_{n-1}}+R_{n-1}(b)e^{-\kappa_{n-1}z_{n-1}})e^{ibx}db=$$

$$=\frac{\sigma_n+i\omega\epsilon_n}{\eta_n^2}\int_{-\infty}^{\infty}\kappa_n G_n(b)e^{-\kappa_n z_{n-1}}e^{ibx}db+\frac{q}{\mu_n\eta_n^2}\int_{-\infty}^{\infty}bR_n(b)e^{-\kappa_n z_{n-1}}e^{ibx}db$$

and

$$\frac{1}{\mu_{n-1}}\int_{-\infty}^{\infty}(Q_{n-1}(b)e^{\kappa_{n-1}z_{n-1}}+R_{n-1}(b)e^{-\kappa_{n-1}z_{n-1}})e^{ibx}db=\frac{1}{\mu_n}\int_{-\infty}^{\infty}R_n(b)e^{-\kappa_n z_{n-1}}e^{ibx}db \tag{4.107}$$

for $z = z_{n-1}$. Equations (4.96)–(4.107) must be satisfied by all values of x; the common factor $e^{i(\omega t-qy)}$ has been divided out from these equations.

As in Section 3.2, the use of formulae (A.4) and (A.6) in equations (4.100)–(4.107) gives

$$\frac{qb}{\eta_j^2} D_j(b) e^{\kappa_j z_j} + \frac{qb}{\eta_j^2} G_j(b) e^{-\kappa_j z_j} + \frac{i\omega\kappa_j}{\eta_j^2} Q_j(b) e^{\kappa_j z_j} - \frac{i\omega\kappa_j}{\eta_j^2} R_j(b) e^{-\kappa_j z_j} + \tag{4.108}$$

$$- \frac{qb}{\eta_j^2} D_{j+1}(b) e^{\kappa_{j+1} z_j} - \frac{qb}{\eta_{j+1}^2} G_{j+1}(b) e^{-\kappa_{j+1} z_j} - \frac{i\omega\kappa_{j+1}}{\eta_{j+1}^2} Q_{j+1}(b) e^{\kappa_{j+1} z_j} +$$

$$+ \frac{i\omega\kappa_{j+1}}{\eta_{j+1}^2} R_{j+1}(b) e^{-\kappa_{j+1} z_j} = 0 ,$$

$$D_j(b) e^{\kappa_j z_j} + G_j(b) e^{-\kappa_j z_j} - D_{j+1}(b) e^{\kappa_{j+1} z_j} - G_{j+1}(b) e^{-\kappa_{j+1} z_j} = 0 , \tag{4.109}$$

$$- \frac{(\sigma_j + i\omega\epsilon_j)\kappa_j}{\eta_j^2} D_j(b) e^{\kappa_j z_j} + \frac{(\sigma_j + i\omega\epsilon_j)\kappa_j}{\eta_j^2} G_j(b) e^{-\kappa_j z_j} + \tag{4.110}$$

$$+ \frac{qb}{\mu_j \eta_j^2} Q_j(b) e^{\kappa_j z_j} + \frac{qb}{\mu_j \eta_j^2} R_j(b) e^{-\kappa_j z_j} + \frac{(\sigma_{j+1} + i\omega\epsilon_{j+1})\kappa_{j+1}}{\eta_{j+1}^2} D_{j+1}(b) e^{\kappa_{j+1} z_j} +$$

$$- \frac{(\sigma_{j+1} + i\omega\epsilon_{j+1})\kappa_{j+1}}{\eta_{j+1}^2} G_{j+1}(b) e^{-\kappa_{j+1} z_j} - \frac{qb}{\mu_{j+1}\eta_{j+1}^2} Q_{j+1}(b) e^{\kappa_{j+1} z_j} +$$

$$- \frac{qb}{\mu_{j+1}\eta_{j+1}^2} R_{j+1}(b) e^{-\kappa_{j+1} z_j} = 0 ,$$

$$\frac{1}{\mu_j} Q_j(b) e^{\kappa_j z_j} + \frac{1}{\mu_j} R_j(b) e^{-\kappa_j z_j} - \frac{1}{\mu_{j+1}} Q_{j+1}(b) e^{\kappa_{j+1} z_j} - \frac{1}{\mu_{j+1}} R_{j+1}(b) e^{-\kappa_{j+1} z_j} = 0 \tag{4.111}$$

(j = 1,..., n−2), and

$$\frac{qb}{\eta_{n-1}^2} D_{n-1}(b) e^{\kappa_{n-1} z_{n-1}} + \frac{qb}{\eta_{n-1}^2} G_{n-1}(b) e^{-\kappa_{n-1} z_{n-1}} + \frac{i\omega\kappa_{n-1}}{\eta_{n-1}^2} Q_{n-1}(b) e^{\kappa_{n-1} z_{n-1}} + \tag{4.112}$$

$$- \frac{i\omega\kappa_{n-1}}{\eta_{n-1}^2} R_{n-1}(b) e^{-\kappa_{n-1} z_{n-1}} - \frac{qb}{\eta_n^2} G_n(b) e^{-\kappa_n z_{n-1}} + \frac{i\omega\kappa_n}{\eta_n^2} R_n(b) e^{-\kappa_n z_{n-1}} = 0 ,$$

$$D_{n-1}(b) e^{\kappa_{n-1} z_{n-1}} + G_{n-1}(b) e^{-\kappa_{n-1} z_{n-1}} - G_n(b) e^{-\kappa_n z_{n-1}} = 0 , \tag{4.113}$$

$$- \frac{(\sigma_{n-1} + i\omega\epsilon_{n-1})\kappa_{n-1}}{\eta_{n-1}^2} D_{n-1}(b)e^{\kappa_{n-1}z_{n-1}} + \tag{4.114}$$

$$+ \frac{(\sigma_{n-1} + i\omega\epsilon_{n-1})\kappa_{n-1}}{\eta_{n-1}^2} G_{n-1}(b)e^{-\kappa_{n-1}z_{n-1}} +$$

$$+ \frac{qb}{\mu_{n-1}\eta_{n-1}^2} Q_{n-1}(b)e^{\kappa_{n-1}z_{n-1}} + \frac{qb}{\mu_{n-1}\eta_{n-1}^2} R_{n-1}(b)e^{-\kappa_{n-1}z_{n-1}} +$$

$$- \frac{(\sigma_n + i\omega\epsilon_n)\kappa_n}{\eta_n^2} G_n(b)e^{-\kappa_n z_{n-1}} - \frac{qb}{\mu_n\eta_n^2} R_n(b)e^{-\kappa_n z_{n-1}} = 0 ,$$

and

$$\frac{1}{\mu_{n-1}} Q_{n-1}(b)e^{\kappa_{n-1}z_{n-1}} + \frac{1}{\mu_{n-1}} R_{n-1}(b)e^{-\kappa_{n-1}z_{n-1}} - \frac{1}{\mu_n} R_n(b)e^{-\kappa_n z_{n-1}} = 0 . \tag{4.115}$$

The set of equations (4.108)–(4.115) valid for every real value of b involves 4n–4 linear equations and 4n–2 unknown coefficients. So two of the coefficients can be considered known, and it is possible to determine the others in terms of the known. Let us regard $G_1(b)$ and $R_1(b)$ as known. These are the only coefficients appearing in the case of a homogeneous earth. So $D_1(b)$ and $Q_1(b)$ can be expressed as

$$D_1(b) = \alpha_G(b)G_1(b) + \alpha_R(b)R_1(b) \tag{4.116}$$

and

$$Q_1(b) = \beta_G(b)G_1(b) + \beta_R(b)R_1(b) . \tag{4.117}$$

The quantities $\alpha_G(b)$, $\alpha_R(b)$, $\beta_G(b)$ and $\beta_R(b)$ depend on the properties of the layers of the earth.

In the case of a homogeneous earth the field is given by equations (4.90)–(4.95) with $n = 1$, so that all factors $\alpha_G(b)$, $\alpha_R(b)$, $\beta_G(b)$ and $\beta_R(b)$ are trivially zero; no equations corresponding to equations (4.100)–(4.107) or (4.108)–(4.115) exist. If the earth consists of two layers, the α- and β-factors are already rather complicated:

$$\alpha_G(b) = \frac{e^{-2\kappa_1 z_1}}{L}\left(-q^2b^2(k_2^2-k_1^2)^2 - k_1^2\kappa_1^2\eta_2^4 + k_2^2\kappa_2^2\eta_1^4 + \right. \tag{4.118}$$

$$\left. - \frac{\mu_2}{\mu_1} k_1^2\kappa_1\kappa_2\eta_1^2\eta_2^2 + \frac{\mu_1}{\mu_2} k_2^2\kappa_1\kappa_2\eta_1^2\eta_2^2\right),$$

$$\alpha_R(b) = \frac{2e^{-2\kappa_1 z_1}}{L} i\omega q b(k_2^2 - k_1^2)\kappa_1\eta_2^2\,, \tag{4.119}$$

$$\beta_G(b) = \frac{2e^{-2\kappa_1 z_1}}{Li\omega} qb(k_2^2 - k_1^2)k_1^2\kappa_1\eta_2^2 \tag{4.120}$$

and

$$\beta_R(b) = \frac{e^{-2\kappa_1 z_1}}{L}\left(-q^2b^2(k_2^2-k_1^2)^2 - k_1^2\kappa_1^2\eta_2^4 + k_2^2\kappa_2^2\eta_1^4 + \right. \tag{4.121}$$

$$\left. + \frac{\mu_2}{\mu_1} k_1^2\kappa_1\kappa_2\eta_1^2\eta_2^2 - \frac{\mu_1}{\mu_2} k_2^2\kappa_1\kappa_2\eta_1^2\eta_2^2\right)$$

where

$$L = q^2b^2(k_2^2-k_1^2)^2 - k_1^2\kappa_1^2\eta_2^4 - k_2^2\kappa_2^2\eta_1^4 - \frac{\mu_2}{\mu_1} k_1^2\kappa_1\kappa_2\eta_1^2\eta_2^2 - \frac{\mu_1}{\mu_2} k_2^2\kappa_1\kappa_2\eta_1^2\eta_2^2 \tag{4.122}$$

As z_1 approaches infinity all α- and β-factors approach zero. This is the expected result, because the value $z_1 = \infty$ corresponds to a homogeneous earth. Another way of achieving a homogeneous earth is to set σ_1, ϵ_1 and μ_1 equal to σ_2, ϵ_2 and μ_2, respectively. This substitution in equations (4.118)–(4.122) also makes all α- and β-factors zero.

From equations (4.84), (4.85), (4.87), (4.88), (4.116) and (4.117) it is seen that the tangential components of the electric and magnetic fields at the earth's surface on its lower side are

$$E_x(x,y,z=0,t) = \int_{-\infty}^{\infty} E_x(b,x,y,t)db\,, \tag{4.123}$$

$$E_y(x,y,z=0,t) = \int_{-\infty}^{\infty} E_y(b,x,y,t)db\,, \tag{4.124}$$

$$B_x(x,y,z=0,t) = \int_{-\infty}^{\infty} B_x(b,x,y,t)db \tag{4.125}$$

and

$$B_y(x, y, z = 0, t) = \int_{-\infty}^{\infty} B_y(b, x, y, t) db \tag{4.126}$$

where

$$E_x(b, x, y, t) = \frac{e^{i(\omega t + bx - qy)}}{\eta_1^2} [(qb(\alpha_G(b) + 1) + i\omega\kappa_1\beta_G(b))G_1(b) + \\ + (qb\alpha_R(b) + i\omega\kappa_1(\beta_R(b) - 1))R_1(b)] \, , \tag{4.127}$$

$$E_y(b, x, y, t) = e^{i(\omega t + bx - qy)}[(\alpha_G(b) + 1)G_1(b) + \alpha_R(b)R_1(b)] \, , \tag{4.128}$$

$$B_x(b, x, y, t) = \frac{e^{i(\omega t + bx - qy)}}{\eta_1^2} [(qb\beta_G(b) - \mu_1(\sigma_1 + i\omega\epsilon_1)\kappa_1(\alpha_G(b) - 1))G_1(b) + \\ + (qb(\beta_R(b) + 1) - \mu_1(\sigma_1 + i\omega\epsilon_1)\kappa_1\alpha_R(b))R_1(b)] \tag{4.129}$$

and

$$B_y(b, x, y, t) = e^{i(\omega t + bx - qy)}[\beta_G(b)G_1(b) + (\beta_R(b) + 1)R_1(b)] \, . \tag{4.130}$$

Using formulae (4.128) and (4.130) equations (4.127) and (4.129) can alternatively be written as

$$E_x(b, x, y, t) = \frac{qb}{\eta_1^2} E_y(b, x, y, t) + \frac{i\omega\kappa_1}{\eta_1^2} B_y(b, x, y, t) - e^{i(\omega t + bx - qy)} \frac{2i\omega\kappa_1 R_1(b)}{\eta_1^2} \tag{4.131}$$

and

$$B_x(b, x, y, t) = \frac{qb}{\eta_1^2} B_y(b, x, y, t) - \frac{\mu_1(\sigma_1 + i\omega\epsilon_1)\kappa_1}{\eta_1^2} E_y(b, x, y, t) + \\ + e^{i(\omega t + bx - qy)} \frac{2\mu_1(\sigma_1 + i\omega\epsilon_1)\kappa_1 G_1(b)}{\eta_1^2} \tag{4.132}$$

At the analogous point in Section 3.2 a surface impedance for each value of b was defined as the quotient of $-E_y$ and $H_x = (1/\mu_1)B_x$ at the earth's surface. However, such a definition in the present connection would not be equally successful, because the »integration constant» functions $G_1(b)$ and $R_1(b)$ would not disappear, and so the surface impedance is not dependent on the properties of the earth only. Of course, this fact does not make it impossible to define two surface impedances:

$$Z_1(b) = -\frac{E_y(b,x,y,t)}{H_x(b,x,y,t)} = -\mu_1 \frac{E_y(b,x,y,t)}{B_x(b,x,y,t)} = \tag{4.133}$$

$$= -\mu_1\eta_1^2((\alpha_G(b)+1)G_1(b) + \alpha_R(b)R_1(b))\big/((qb\beta_G(b) +$$

$$-\mu_1(\sigma_1 + i\omega\epsilon_1)\kappa_1(\alpha_G(b)-1))G_1(b) + (qb(\beta_R(b)+1) +$$

$$-\mu_1(\sigma_1 + i\omega\epsilon_1)\kappa_1\alpha_R(b))R_1(b))$$

and

$$Z_2(b) = \frac{E_x(b,x,y,t)}{H_y(b,x,y,t)} = \mu_1 \frac{E_x(b,x,y,t)}{B_y(b,x,y,t)} = \tag{4.134}$$

$$= \frac{\mu_1((qb(\alpha_G(b)+1) + i\omega\kappa_1\beta_G(b))G_1(b) + (qb\alpha_R(b) + i\omega\kappa_1(\beta_R(b)-1))R_1(b))}{\eta_1^2(\beta_G(b)G_1(b) + (\beta_R(b)+1)R_1(b))} .$$

If the earth is homogeneous, in which case the electromagnetic field within the earth is described by formulae (4.90)–(4.95) with n = 1, the impedances are

$$Z_1(b) = -\frac{\mu_1\eta_1^2 G_1(b)}{\mu_1(\sigma_1 + i\omega\epsilon_1)\kappa_1 G_1(b) + qbR_1(b)} \tag{4.135}$$

and

$$Z_2(b) = \frac{\mu_1(qbG_1(b) - i\omega\kappa_1 R_1(b))}{\eta_1^2 R_1(b)} \tag{4.136}$$

These are also obtained with the substitution $\alpha_G(b) = \alpha_R(b) = \beta_G(b) = \beta_R(b) = 0$ into formulae (4.133) and (4.134).

If q is equal to zero, equations (4.108)–(4.115) are resolved into two sets of equations, one containing only $D(b)$- and $G(b)$-coefficients and the other only $Q(b)$- and $R(b)$-coefficients. Then $\alpha_R(b)$ and $\beta_G(b)$ are necessarily zero. Substitution of $q = \alpha_R(b) = \beta_G(b) = 0$ into formulae (4.133) and (4.134) yields

$$Z_1(b) = -\frac{i\omega\mu_1}{\kappa_1}\frac{\alpha_G(b)+1}{\alpha_G(b)-1} \tag{4.137}$$

and

$$Z_2(b) = -\frac{\kappa_1}{\sigma_1 + i\omega\epsilon_1}\frac{\beta_R(b)-1}{\beta_R(b)+1} \tag{4.138}$$

where equations (4.80) and (B.41) (with subscript 1) have been utilized. Now the integration constants have vanished from the expressions of the impedances, and the latter throw light on the properties of the earth. Equations (4.137) and (3.51), as well as equations (4.138) and (3.53), are the same ($\alpha_G(b) = \alpha_1(b)$ and $\beta_R(b) = \alpha_1'(b)$).

Using formulae (A.4) and (A.6) equations (4.96)–(4.99) give

$$\frac{qb}{\eta^2}D_0(b) + \frac{i\omega\kappa_0}{\eta^2}Q_0(b) - \frac{qb}{\eta_1^2}D_1(b) - \frac{qb}{\eta_1^2}G_1(b) + \tag{4.139}$$

$$-\frac{i\omega\kappa_1}{\eta_1^2}Q_1(b) + \frac{i\omega\kappa_1}{\eta_1^2}R_1(b) = \frac{i\omega\mu_0 qbJe^{-\kappa_0 h}}{4\pi k_0^2\kappa_0},$$

$$D_0(b) - D_1(b) - G_1(b) = \frac{i\omega\mu_0\eta^2 Je^{-\kappa_0 h}}{4\pi k_0^2\kappa_0}, \tag{4.140}$$

$$-\frac{(\sigma_0 + i\omega\epsilon_0)\kappa_0}{\eta^2}D_0(b) + \frac{qb}{\mu_0\eta^2}Q_0(b) + \frac{(\sigma_1 + i\omega\epsilon_1)\kappa_1}{\eta_1^2}D_1(b) + \tag{4.141}$$

$$-\frac{(\sigma_1 + i\omega\epsilon_1)\kappa_1}{\eta_1^2}G_1(b) - \frac{qb}{\mu_1\eta^2}Q_1(b) - \frac{qb}{\mu_1\eta_1^2}R_1(b) = -\frac{J}{4\pi}e^{-\kappa_0 h}$$

and

$$\frac{1}{\mu_0}Q_0(b) - \frac{1}{\mu_1}Q_1(b) - \frac{1}{\mu_1}R_1(b) = 0\,. \tag{4.142}$$

Equations (A.72) and (A.75) have also been employed here. Before using formula (A.72) in the derivation of equation (4.139) it is necessary to take equation (A.40) into account and to perform a partial integration for the first term of equation (4.96).

Formulae (4.116) and (4.117) can now be substituted into equations (4.139)–(4.142), and so four linear inhomogeneous equations are obtained. From them the unknown coefficients $D_0(b)$, $Q_0(b)$, $G_1(b)$ and $R_1(b)$ can be calculated as functions of the properties of the layered earth, of the conductivity of the air, and of the height, the frequency, the longitudinal propagation constant, the magnitude and the phase of the primary current. Then $D_0(b)$ and $Q_0(b)$ are expressible as

$$D_0(b) = \alpha_0(b) \frac{i\omega\mu_0\eta^2 J e^{-\kappa_0 h}}{4\pi k_0^2 \kappa_0} \tag{4.143}$$

and

$$Q_0(b) = \beta_0(b) \frac{\mu_0 J e^{-\kappa_0 h}}{2\pi} . \tag{4.144}$$

The explicit determination of $\alpha_0(b)$ and $\beta_0(b)$ would be straightforward but laborious and is neglected. Their expressions are implicitly involved in Section 4.5 as will be indicated. The reason for the somewhat complicated way of expressing $D_0(b)$ and $Q_0(b)$ in equations (4.143) and (4.144) also becomes clear in Section 4.5.

The formulae for the total electromagnetic field $\bar{E}_M(x, y, t)$, $\bar{B}_M(x, y, t)$ at the earth's surface on its upper side can now be written: $\bar{E}_M(x, y, t)$ is equal to the sum of the field of equation (4.59) and of the field given by formulae (4.64), (4.76), (4.77), (4.143) and (4.144) with $z = 0$, and $\bar{B}_M(x, y, t)$ is equal to the sum of the field of equation (4.60) and of the field expressed in formulae (4.75), (4.78), (4.79), (4.143) and (4.144) with $z = 0$. The explicit equations for $\bar{E}_M$ and $\bar{B}_M$, as well as for the potential difference $U_{P_1P_2}(t)$ between two points P_1 and P_2 on the earth's surface, are now omitted, because they are also implicitly included in equations in Section 4.5.

Owing to the continuity of the tangential components of the fields $\bar{E}$ and $\bar{H}$ the x- and y-components of $\bar{E}_M$ and $(\mu_1/\mu_0)\bar{B}_M$ are equal to $E_{x,y}(x, y, z = 0, t)$ and $B_{x,y}(x, y, z = 0, t)$ given by formulae (4.123)–(4.130). The normal component of $\bar{B}$ is continuous (formula (B.53)), so that B_z of equation (4.89) with $z = 0$ is equal to B_{Mz}. The investigation of E_{Mz} and E_z of equation (4.86) with $z = 0$ gives information on charges appearing at the earth's surface (see formulae (B.52) and (B.62)).

4.5 Induction in an earth having arbitrarily changing properties in the vertical and »almost arbitrarily» changing properties in the transverse horizontal direction

Let us allow the conductivity σ, the permittivity ϵ and the permeability μ of the earth be any functions of the depth z and the transverse horizontal coordinate x. After a while a restriction to this x-dependence will be made. The primary electromagnetic field is still expressed by formulae (4.59) and (4.60) (with Imq = 0 and $\sigma_0 > 0$). Since the properties of the earth are independent of y, all y-dependence is given by e^{-iqy}. The time-dependence is $e^{i\omega t}$. So the calculations of the secondary field in the air presented in Section 4.4 is valid, again, and the secondary field is thus described by equations (4.64) and (4.75)–(4.79).

As in Sections 2.7 and 3.4, the electromagnetic field within the earth satisfies Maxwell's equations (2.93)–(2.96). The component forms of formulae (2.93) and (2.94) are similar to equations (4.65)–(4.70): but μH has to be written in the place of B in formulae (4.65)–(4.67) and H in place of B in formulae (4.68)–(4.70), and in the latter μ_0 and the subscript 0 must be removed. The equations analogous to formulae (4.71)–(4.74) are then

$$E_x = -\frac{iq\dfrac{\partial E_y}{\partial x} - i\omega\mu\dfrac{\partial H_y}{\partial z}}{\eta^2}\,, \tag{4.145}$$

$$E_z = -\frac{iq\dfrac{\partial E_y}{\partial z} + i\omega\mu\dfrac{\partial H_y}{\partial x}}{\eta^2}\,, \tag{4.146}$$

$$H_x = -\frac{(\sigma + i\omega\epsilon)\dfrac{\partial E_y}{\partial z} + iq\dfrac{\partial H_y}{\partial x}}{\eta^2} \tag{4.147}$$

and

$$H_z = \frac{(\sigma + i\omega\epsilon)\dfrac{\partial E_y}{\partial x} - iq\dfrac{\partial H_y}{\partial x}}{\eta^2} \tag{4.148}$$

The quantity η, which now depends on x and z, has a similar definition as above:

$$\eta^2 = \eta(x,z)^2 = k(x,z)^2 - q^2 \tag{4.149}$$

and

$$-\frac{\pi}{2} < \arg\eta(x,z) \leqslant \frac{\pi}{2}. \tag{4.150}$$

where $k(x, z)$ is the space-dependent propagation constant of the earth. Let us again assume that the conductivity of the earth is everywhere non-zero. Then $\eta(x, z)$ differs from zero at every point of the earth and $-\pi/2 < \arg\eta(x, z) < 0$. Notice that the symbol η had earlier (formulae (4.8) and (4.9)) another meaning than in equations (4.145)–(4.148). Confusion is, however, avoided, when the arguments x and z are explicitly written in the new η and later the value of $\eta(x,z)$ at the earth's surface will be symbolized by η_1.

Equations (4.145)–(4.148) show that after E_y and H_y have been calculated the other field components can be obtained in terms of these, as in Section 4.4. Analogously to Sections 2.7 and 3.4, E_y and H_y are to be determined from the wave equations (2.98) and (2.99) also utilizing the facts that σ, ϵ and μ are independent of y and that the y-dependence of the fields is e^{-iqy}. Substituting formulae (4.145)–(4.148) into the y-components of equations (2.98) and (2.99) we obtain:

$$\begin{aligned}
&\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + \eta^2 E_y + \frac{iq}{\sigma + i\omega\epsilon}\left[\frac{\partial(\sigma + i\omega\epsilon)}{\partial x}\left(\frac{iq}{\eta^2}\frac{\partial E_y}{\partial x} - \frac{i\omega\mu}{\eta^2}\frac{\partial H_y}{\partial z}\right) + \right.\\
&\left. + \frac{\partial(\sigma + i\omega\epsilon)}{\partial z}\left(\frac{iq}{\eta^2}\frac{\partial E_y}{\partial z} + \frac{i\omega\mu}{\eta^2}\frac{\partial H_y}{\partial x}\right)\right] - \frac{1}{\mu}\left[\frac{\partial\mu}{\partial x}\left(\frac{k^2}{\eta^2}\frac{\partial E_y}{\partial x} - \frac{\omega\mu q}{\eta^2}\frac{\partial H_y}{\partial z}\right) + \right.\\
&\left. + \frac{\partial\mu}{\partial z}\left(\frac{k^2}{\eta^2}\frac{\partial E_y}{\partial z} + \frac{\omega\mu q}{\eta^2}\frac{\partial H_y}{\partial x}\right)\right] = 0
\end{aligned} \tag{4.151}$$

and

$$\begin{aligned}
&\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} + \eta^2 H_y + \frac{iq}{\mu}\left[\frac{\partial\mu}{\partial x}\left(\frac{iq}{\eta^2}\frac{\partial H_y}{\partial x} + \frac{\sigma + i\omega\epsilon}{\eta^2}\frac{\partial E_y}{\partial z}\right) + \right.\\
&\left. + \frac{\partial\mu}{\partial z}\left(\frac{iq}{\eta^2}\frac{\partial H_y}{\partial z} - \frac{\sigma + i\omega\epsilon}{\eta^2}\frac{\partial E_y}{\partial x}\right)\right] - \frac{1}{\sigma + i\omega\epsilon}\left[\frac{\partial(\sigma + i\omega\epsilon)}{\partial x}\left(\frac{k^2}{\eta^2}\frac{\partial H_y}{\partial x} + \right.\right.\\
&\left.\left. - \frac{i(\sigma + i\omega\epsilon)q}{\eta^2}\frac{\partial E_y}{\partial z}\right) + \frac{\partial(\sigma + i\omega\epsilon)}{\partial z}\left(\frac{k^2}{\eta^2}\frac{\partial H_y}{\partial z} + \frac{i(\sigma + i\omega\epsilon)q}{\eta^2}\frac{\partial E_y}{\partial x}\right)\right] = 0.
\end{aligned} \tag{4.152}$$

Equations (4.151) and (4.152) are partial differential equations of the second order which are also coupled to each other in the sense that both contain E_y and

H_y. Neither these equations nor their solutions are treated in more detail in this study. Let us merely denote the most general physically acceptable x- and z-depending parts of E_y and H_y by $f(x, z)$ and $g(x, z)$, respectively, which thus satisfy equations (4.151) and (4.152). Hence

$$E_y(x, y, z, t) = f(x, z)e^{i(\omega t - qy)} \tag{4.153}$$

and

$$H_y(x, y, z, t) = g(x, z)e^{i(\omega t - qy)} \; . \tag{4.154}$$

Expressing $f(x, z)$ and $g(x, z)$ as Fourier integrals with respect to x, equations (4.153) and (4.154) can be written as

$$E_y(x, y, z, t) = e^{i(\omega t - qy)} \int_{-\infty}^{\infty} f(b, z)e^{ibx}db \tag{4.155}$$

and

$$H_y(x, y, z, t) = e^{i(\omega t - qy)} \int_{-\infty}^{\infty} g(b, z)e^{ibx}db \tag{4.156}$$

assuming that the Fourier transforms and the inverse Fourier transforms in question exist. From equations (4.145)–(4.148), (4.155) and (4.156) the following equations are obtained:

$$E_x(x, y, z, t) = \frac{e^{i(\omega t - qy)}}{\eta(x, z)^2}\left[q \int_{-\infty}^{\infty} bf(b, z)e^{ibx}db + i\omega\mu(x, z) \int_{-\infty}^{\infty} \frac{\partial g(b, z)}{\partial z} e^{ibx}db \right], \tag{4.157}$$

$$E_z(x, y, z, t) = \frac{e^{i(\omega t - qy)}}{\eta(x, z)^2}\left[-iq \int_{-\infty}^{\infty} \frac{\partial f(b, z)}{\partial z} e^{ibx}db + \omega\mu(x, z) \int_{-\infty}^{\infty} bg(b, z)e^{ibx}db \right], \tag{4.158}$$

$$H_x(x, y, z, t) = \frac{e^{i(\omega t - qy)}}{\eta(x, z)^2}\left[-(\sigma(x, z) + i\omega\epsilon(x, z)) \int_{-\infty}^{\infty} \frac{\partial f(b, z)}{\partial z} e^{ibx}db + \right. \tag{4.159}$$

$$\left. + q \int_{-\infty}^{\infty} bg(b, z)e^{ibx}db \right]$$

and

$$H_z(x, y, z, t) = \frac{e^{i(\omega t - qy)}}{\eta(x, z)^2}\left[i(\sigma(x, z) + i\omega\epsilon(x, z)) \int_{-\infty}^{\infty} bf(b, z)e^{ibx}db + \right. \tag{4.160}$$

$$\left. - iq \int_{-\infty}^{\infty} \frac{\partial g(b, z)}{\partial z} e^{ibx}db \right].$$

The dependence of σ, ϵ, μ and η on x and z is explicitly stated in these formulae.

The electromagnetic field in the earth expressed by formulae (4.155)–(4.160) satisfies Maxwell's equations (2.93)–(2.96): The validity of the x- and z-components of formulae (2.93) and (2.94) is involved in the calculation of E_x, E_z, H_x, and H_z. Their y-components are also satisfied owing to the assumption of the validity equations (4.151) and (4.152) with formulae (4.153) and (4.154). Equations (2.95) and (2.96) are consequences of formulae (2.93) and (2.94), respectively. The right-hand side of equation (2.96) expresses the volume charge density in the earth, which in the present case is generally not zero.

As in Sections 2.7 and 3.4, the dependence of μ and $\sigma + i\omega\epsilon$ on x and z should be known in order that the functions $f(b, z)$ and $g(b, z)$ could be investigated thoroughly. So all discussion of the directions of attenuation and of phase and energy propagation with respect to z is neglected here. (The comments presented in Section 4.4 on the x- and y-dependencies are valid.)

The earth may contain discontinuities in its electromagnetic properties. The solution expressed by equations (4.155)–(4.160) must then be calculated in each continuous region separately and boundary conditions have to be used. Let us assume that all such possible discontinuities have already been taken into account and equations (4.155)–(4.160) represent the field in the whole earth. Equations (4.84)–(4.95) are a special case of formulae (4.155)–(4.160). The former were obtained using the separation of variables, but the latter were derived employing the Fourier transform (*cf.* Chapter 3).

We assume that the conductivity is not infinite in the uppermost part of the earth. Then the tangential component of $\bar{H}$ in addition to that of $\bar{E}$ is continuous at the earth's surface. This yields boundary conditions identical with equations (4.96)–(4.99), when the right-hand sides of these formulae are replaced by the right-hand sides of equations (4.157), (4.155), (4.159) and (4.156) with $z = 0$ and the factor $e^{i(\omega t - qy)}$ is omitted. If these equations are treated in the same manner as formulae (4.139)–(4.142) were derived from equations (4.96)–(4.99), it is seen that the dependence of the electrical parameters of the earth on x at the earth's surface results in different values of b being coupled to each other (*cf.* equation (A.13)). In order to obtain a separate set of four equations for each value of b, let us assume that the electrical parameters of the earth are independent of x at the earth's surface. Then using formulae (A.4) and (A.6) the following equations are obtained:

$$\frac{qb}{\eta^2} D_0(b) + \frac{i\omega\kappa_0}{\eta^2} Q_0(b) - \frac{qb}{\eta_1^2} f(b) - \frac{i\omega\mu_1}{\eta_1^2} g'(b) = \frac{i\omega\mu_0 qbJe^{-\kappa_0 h}}{4\pi k_0^2 \kappa_0}, \qquad (4.161)$$

$$D_0(b) - f(b) = \frac{i\omega\mu_0\eta^2 J e^{-\kappa_0 h}}{4\pi k_0^2 \kappa_0} \,, \tag{4.162}$$

$$-\frac{(\sigma_0 + i\omega\epsilon_0)\kappa_0}{\eta^2} D_0(b) + \frac{qb}{\mu_0\eta^2} Q_0(b) + \frac{\sigma_1 + i\omega\epsilon_1}{\eta_1^2} f'(b) - \frac{qb}{\eta_1^2} g(b) = -\frac{J}{4\pi} e^{-\kappa_0 h} \tag{4.163}$$

and

$$\frac{1}{\mu_0} Q_0(b) - g(b) = 0 \,. \tag{4.164}$$

In these equations, which are valid for every real value of b, $f(b, z = 0)$ is denoted by $f(b)$, $g(b, z = 0)$ by $g(b)$, $\partial f(b, z)/\partial z]_{z=0}$ by $f'(b)$, $\partial g(b, z)/\partial z]_{z=0}$ by $g'(b)$, $\sigma(x, z = 0)$ by σ_1, $\epsilon(x, z = 0)$ by ϵ_1, $\mu(x, z = 0)$ by μ_1 and $\eta(x, z = 0)$ by η_1. According to the assumption made above σ_1, ϵ_1, μ_1 and η_1 are independent of x.

The four linear equations (4.161)–(4.164) may seem to contain six unknown quantities $D_0(b)$, $Q_0(b)$, $f(b)$, $g(b)$, $f'(b)$ and $g'(b)$, and so the number of equations would not be sufficient. However, consideration of equations (4.139)–(4.142) associated with a layered earth and of formulae (4.116) and (4.117) suggests that $f(b)$, $g(b)$, $f'(b)$ and $g'(b)$ include only two unknowns, in the case of a layered earth $G_1(b)$ and $R_1(b)$. Furthermore, if the problem is supposed to have an unambiguous solution, *i.e.* the total electromagnetic field can be determined as a function of the primary source and the properties of the earth, equations (4.161)–(4.164) must have a single-valued solution. Thus actually only two unknown quantities can appear in $f(b)$, $g(b)$, $f'(b)$ and $g'(b)$. But in order to be able to obtain a solution to the problem, the dependence of the f- and g-quantities on the unknowns must be known precisely.

Let us assume that the original unknown quantities can be expressed in terms of $f(b)$ and $g(b)$, which are now considered new unknown quantities. Assume further that the dependence of $f'(b)$ and $g'(b)$ on $f(b)$ and $g(b)$ obtained by substituting the expressions of the original unknowns is linear with known coefficients $c_f(b)$, $c_g(b)$, $m_f(b)$ and $m_g(b)$:

$$f'(b) = c_f(b) f(b) + c_g(b) g(b) \tag{4.165}$$

and

$$g'(b) = m_f(b) f(b) + m_g(b) g(b) \,. \tag{4.166}$$

In the case of a layered earth the f- and g-quantities and the c- and m-coefficients are

$$f(b) = D_1(b) + G_1(b)\,, \tag{4.167}$$

$$f'(b) = \kappa_1 D_1(b) - \kappa_1 G_1(b)\,, \tag{4.168}$$

$$g(b) = \frac{1}{\mu_1}\,(Q_1(b) + R_1(b))\,, \tag{4.169}$$

$$g'(b) = \frac{1}{\mu_1}\,(\kappa_1 Q_1(b) - \kappa_1 R_1(b))\,, \tag{4.170}$$

$$c_f(b) = \frac{\kappa_1(\alpha_G(b)\beta_R(b) - \alpha_R(b)\beta_G(b) + \alpha_G(b) - \beta_R(b) - 1)}{\alpha_G(b)\beta_R(b) - \alpha_R(b)\beta_G(b) + \alpha_G(b) + \beta_R(b) + 1}\,, \tag{4.171}$$

$$c_g(b) = \frac{2\mu_1\kappa_1\alpha_R(b)}{\alpha_G(b)\beta_R(b) - \alpha_R(b)\beta_G(b) + \alpha_G(b) + \beta_R(b) + 1}\,, \tag{4.172}$$

$$m_f(b) = \frac{2\kappa_1\beta_G(b)}{\mu_1(\alpha_G(b)\beta_R(b) - \alpha_R(b)\beta_G(b) + \alpha_G(b) + \beta_R(b) + 1)} \tag{4.173}$$

and

$$m_g(b) = \frac{\kappa_1(\alpha_G(b)\beta_R(b) - \alpha_R(b)\beta_G(b) - \alpha_G(b) + \beta_R(b) - 1)}{\alpha_G(b)\beta_R(b) - \alpha_R(b)\beta_G(b) + \alpha_G(b) + \beta_R(b) + 1}\,. \tag{4.174}$$

These formulae are obtained using equations (4.85), (4.88), (4.116) and (4.117). In the case of a homogeneous earth, where $D_1(b)$ and $Q_1(b)$ are zero, the coefficients are simply $c_f(b) = m_g(b) = -\kappa_1$ and $c_g(b) = m_f(b) = 0$.

The question of the existence of linear equations (4.165) and (4.166) such that the coefficients $c_f(b)$, $c_g(b)$, $m_f(b)$ and $m_g(b)$ are really independent of the original and new unknown quantities, *i.e.* known for each b as functions of the properties of the earth and q and ω, is not discussed in this work. The discussion would require consideration of equations (4.151) and (4.152) determining $f(x, z)$ and $g(x, z)$. Anyway, the linearity of these partial differential equations suggests the existence of equations (4.165) and (4.166). The treatment which follows is valid for any coefficients $c_f(b)$, $c_g(b)$, $m_f(b)$ and $m_g(b)$ in formulae (4.165) and (4.166). Thus the coefficients may in principle be functions of any unknown quantities. In the simplest case $c_f(b) = f'(b)/f(b)$, $m_g(b) = g'(b)/g(b)$ and $c_g(b) = m_f(b) = 0$. However, the value of the results is questionable if the coefficients include unknown quantities.

Substitution of formulae (4.165) and (4.166) in equations (4.161)–(4.164) yields

$$\frac{qb}{\eta^2} D_0(b) + \frac{i\omega\kappa_0}{\eta^2} Q_0(b) - \left(\frac{qb}{\eta_1^2} + \frac{i\omega\mu_1 m_f(b)}{\eta_1^2}\right) f(b) - \frac{i\omega\mu_1 m_g(b)}{\eta_1^2} g(b) = \tag{4.175}$$
$$= \frac{i\omega\mu_0 qbJe^{-\kappa_0 h}}{4\pi k_0^2 \kappa_0} \; ,$$

$$D_0(b) - f(b) = \frac{i\omega\mu_0\eta^2 Je^{-\kappa_0 h}}{4\pi k_0^2\kappa_0} \; , \tag{4.176}$$

$$-\frac{(\sigma_0 + i\omega\epsilon_0)\kappa_0}{\eta^2} D_0(b) + \frac{qb}{\mu_0\eta^2} Q_0(b) + \frac{(\sigma_1 + i\omega\epsilon_1)c_f(b)}{\eta_1^2} f(b) + \tag{4.177}$$
$$- \left(\frac{qb}{\eta_1^2} - \frac{(\sigma_1 + i\omega\epsilon_1)c_g(b)}{\eta_1^2}\right) g(b) = -\frac{Je^{-\kappa_0 h}}{4\pi}$$

and

$$\frac{1}{\mu_0} Q_0(b) - g(b) = 0 \; . \tag{4.178}$$

The aim of these considerations is to obtain the expressions of the electromagnetic field on the earth's surface. So it is not necessary to solve all four unknown quantities $D_0(b)$, $Q_0(b)$, $f(b)$ and $g(b)$ from equations (4.175)–(4.178); $D_0(b)$ and $Q_0(b)$ are enough. Solving these from the four linear equations (4.175)–(4.178) gives

$$D_0(b) = \frac{i\omega\mu_0\eta^2 Je^{-\kappa_0(b)h}}{4\pi k_0^2\kappa_0(b)} \; \frac{R_1(b) - R_2(b)}{R_1(b) + R_2(b)} \tag{4.179}$$

and

$$Q_0(b) = \frac{\mu_0 Je^{-\kappa_0(b)h}}{2\pi} \; \frac{R_3(b)}{R_1(b) + R_2(b)} \tag{4.180}$$

The following abbreviations are used in these formulae:

$$R_1(b) = q^2b^2(k_1^2 - k_0^2)^2 - \frac{c_g(b)}{i\omega\mu_1} qbk_1^2(k_1^2 - k_0^2)\eta^2 - i\omega\mu_1 m_f(b)qb(k_1^2 - k_0^2)\eta^2 +$$
$$+ k_1^2\eta^4(m_f(b)c_g(b) - c_f(b)m_g(b)) + \frac{\mu_0}{\mu_1} k_1^2\eta^2\eta_1^2\kappa_0(b)c_f(b) \; , \tag{4.181}$$

$$R_2(b) = -k_0^2\eta_1^4\kappa_0(b)^2 + \frac{\mu_1}{\mu_0} k_0^2\eta^2\eta_1^2\kappa_0(b)m_g(b) \tag{4.182}$$

and

$$R_3(b) = \eta^2\eta_1^2(i\omega\mu_1 m_f(b)\eta^2 - qb(k_1^2 - k_0^2)) \; . \tag{4.183}$$

Here k_1 is, of course, equal to $k(x, z = 0)$, which does not depend on x. The dependence of κ_0 on b (equation (4.62)) is explicitly recognized in formulae (4.179)–(4.183). Substitution of equations (4.171)–(4.174) in formulae (4.181)–(4.183) yields the quantities, $R_1(b)$, $R_2(b)$ and $R_3(b)$ for a layered earth. Comparison of equations (4.143) and (4.179) and of equations (4.144) and (4.180) gives expressions for $\alpha_0(b)$ and $\beta_0(b)$, which are equal to $(R_1(b) - R_2(b))/(R_1(b) + R_2(b))$ and $R_3(b)/(R_1(b) + R_2(b))$, respectively. The use of the »complicated» formulae (4.143) and (4.144) is also now justified.

Using equations (4.59), (4.60), (4.64), (4.75)–(4.79), (4.179) and (4.180) the electromagnetic field at the earth's surface on its upper side has the following expressions:

$$\begin{aligned} E_{Mx}(x, y, t) &= -\frac{i\omega\mu_0\eta qxJe^{i(\omega t-qy)}}{4k_0^2\sqrt{x^2+h^2}} H_1^{(2)}(\eta\sqrt{x^2+h^2}) + \\ &+ \frac{i\omega\mu_0 qJe^{i(\omega t-qy)}}{4\pi k_0^2} \int_{-\infty}^{\infty} \frac{be^{-\kappa_0 h}(R_1 - R_2)}{\kappa_0(R_1 + R_2)} e^{ibx}\, db + \\ &+ \frac{i\omega\mu_0 Je^{i(\omega t-qy)}}{2\pi\eta^2} \int_{-\infty}^{\infty} \frac{\kappa_0 e^{-\kappa_0 h}R_3}{R_1 + R_2} e^{ibx}\, db = \\ &= -\frac{i\omega\mu_0 Je^{i(\omega t-qy)}}{2\pi k_0^2\eta^2} \int_{-\infty}^{\infty} \frac{e^{-\kappa_0 h}(qb\eta^2 R_2 - k_0^2\kappa_0^2 R_3)}{\kappa_0(R_1 + R_2)} e^{ibx}\, db \; , \end{aligned} \tag{4.184}$$

$$\begin{aligned} E_{My}(x, y, t) &= -\frac{\omega\mu_0\eta^2 Je^{i(\omega t-qy)}}{4k_0^2} H_0^{(2)}(\eta\sqrt{x^2+h^2}) + \\ &+ \frac{i\omega\mu_0\eta^2 Je^{i(\omega t-qy)}}{4\pi k_0^2} \int_{-\infty}^{\infty} \frac{e^{-\kappa_0 h}(R_1 - R_2)}{\kappa_0(R_1 + R_2)} e^{ibx}\, db = \\ &= -\frac{i\omega\mu_0\eta^2 Je^{i(\omega t-qy)}}{2\pi k_0^2} \int_{-\infty}^{\infty} \frac{e^{-\kappa_0 h}R_2}{\kappa_0(R_1 + R_2)} e^{ibx}\, db \; , \end{aligned} \tag{4.185}$$

$$E_{Mz}(x,y,t) = -\frac{i\omega\mu_0\eta q h J e^{i(\omega t-qy)}}{4k_0^2\sqrt{x^2+h^2}} H_1^{(2)}(\eta\sqrt{x^2+h^2}) + \tag{4.186}$$

$$+ \frac{\omega\mu_0 q J e^{i(\omega t-qy)}}{4\pi k_0^2}\int_{-\infty}^{\infty}\frac{e^{-\kappa_0 h}(R_1-R_2)}{R_1+R_2}e^{ibx}db +$$

$$+ \frac{\omega\mu_0 J e^{i(\omega t-qy)}}{2\pi\eta^2}\int_{-\infty}^{\infty}\frac{be^{-\kappa_0 h}R_3}{R_1+R_2}e^{ibx}db =$$

$$= \frac{\omega\mu_0 J e^{i(\omega t-qy)}}{2\pi k_0^2\eta^2}\int_{-\infty}^{\infty}\frac{e^{-\kappa_0 h}(q\eta^2 R_1 + bk_0^2 R_3)}{R_1+R_2}e^{ibx}db,$$

$$B_{Mx}(x,y,t) = -\frac{i\mu_0\eta h J e^{i(\omega t-qy)}}{4\sqrt{x^2+h^2}} H_1^{(2)}(\eta\sqrt{x^2+h^2}) + \tag{4.187}$$

$$+ \frac{\mu_0 J e^{i(\omega t-qy)}}{4\pi}\int_{-\infty}^{\infty}\frac{e^{-\kappa_0 h}(R_1-R_2)}{R_1+R_2}e^{ibx}db +$$

$$+ \frac{\mu_0 q J e^{i(\omega t-qy)}}{2\pi\eta^2}\int_{-\infty}^{\infty}\frac{be^{-\kappa_0 h}R_3}{R_1+R_2}e^{ibx}db =$$

$$= \frac{\mu_0 J e^{i(\omega t-qy)}}{2\pi\eta^2}\int_{-\infty}^{\infty}\frac{e^{-\kappa_0 h}(\eta^2 R_1 + qbR_3)}{R_1+R_2}e^{ibx}db,$$

$$B_{My}(x,y,t) = \frac{\mu_0 J e^{i(\omega t-qy)}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-\kappa_0 h}R_3}{R_1+R_2}e^{ibx}db \tag{4.188}$$

and

$$B_{Mz}(x,y,t) = \frac{i\mu_0\eta x J e^{i(\omega t-qy)}}{4\sqrt{x^2+h^2}} H_1^{(2)}(\eta\sqrt{x^2+h^2}) + \tag{4.189}$$

$$- \frac{i\mu_0 J e^{i(\omega t-qy)}}{4\pi}\int_{-\infty}^{\infty}\frac{be^{-\kappa_0 h}(R_1-R_2)}{\kappa_0(R_1+R_2)}e^{ibx}db +$$

$$- \frac{i\mu_0 q J e^{i(\omega t-qy)}}{2\pi\eta^2}\int_{-\infty}^{\infty}\frac{\kappa_0 e^{-\kappa_0 h}R_3}{R_1+R_2}e^{ibx}db =$$

$$= \frac{i\mu_0 J e^{i(\omega t-qy)}}{2\pi\eta^2}\int_{-\infty}^{\infty}\frac{e^{-\kappa_0 h}(b\eta^2 R_2 - q\kappa_0^2 R_3)}{\kappa_0(R_1+R_2)}e^{ibx}db.$$

Formulae (A.40), (A.74) and (A.76) were used when expressing the primary electromagnetic field as an integral in equations (4.184)–(4.189) (*cf.* formulae (3.57)–(3.59)).

Utilizing equations (4.184) and (4.185) and the convention concerning the path of integration and made in Chapter 1 the potential difference $U_{P_1P_2}(t)$ between two points $P_1 = (x_1, y_1, 0)$ and $P_2 = (x_2, y_2, 0)$ on the earth's surface is

$$U_{P_1P_2}(t) = \int_{P_1 \text{ straight line}}^{P_2} \bar{E} \cdot d\bar{l} = \int_{P_1 \text{ s.l.}}^{P_2} \bar{E}_M \cdot d\bar{l} = \int_{P_1 \text{ s.l.}}^{P_2} E_{Mx}(x,y,t)dx + E_{My}(x,y,t)dy = \tag{4.190}$$

$$= -\frac{\omega\mu_0 J e^{i\omega t}}{2\pi k_0^2 \eta^2} \int_{-\infty}^{\infty} \left[\frac{(e^{i(bx_2 - qy_2)} - e^{i(bx_1 - qy_1)})(x_2 - x_1)(qb\eta^2 R_2 - k_0^2 \kappa_0^2 R_3) e^{-\kappa_0 h}}{\kappa_0 (b(x_2 - x_1) - q(y_2 - y_1))(R_1 + R_2)} + \right.$$

$$\left. + \frac{(e^{i(bx_2 - qy_2)} - e^{i(bx_1 - qy_1)})(y_2 - y_1)\eta^4 R_2 e^{-\kappa_0 h}}{\kappa_0 (b(x_2 - x_1) - q(y_2 - y_1))(R_1 + R_2)} \right] db \, .$$

As in Sections 2.2 and 3.2, $U_{P_1P_2}$ defined in the above manner describes the potential drop from P_1 to P_2. The constant J connected with the primary source can be solved in terms of a magnetic field component on the earth's surface from one of equations (4.187)–(4.189), and thus the potential difference $U_{P_1P_2}$ is expressible by means of the variation in the geomagnetic field observed on the earth's surface (*cf.* the end of Section 3.2 and formula (2.36)). If the potential drop is considered along a line parallel to the primary source current, *i.e.* x_1 and x_2 are equal, the result is

$$U_{P_1P_2}(t) = \int_{y_1}^{y_2} E_{My}(x,y,t)dy = \frac{\omega\mu_0 \eta^2 J e^{i\omega t}(e^{-iqy_2} - e^{-iqy_1})}{2\pi k_0^2 q} \int_{-\infty}^{\infty} \frac{e^{-\kappa_0 h} R_2}{\kappa_0 (R_1 + R_2)} e^{ibx} db \tag{4.191}$$

where x is the common value of x_1 and x_2.

If equations (4.171)–(4.174) are substituted in formulae (4.181)–(4.183) and R_1, R_2 and R_3 then into equations (4.184)–(4.190), expressions for $\bar{E}_M(x,y,t)$, $\bar{B}_M(x,y,t)$ and $U_{P_1P_2}(t)$ in the case of a layered earth are obtained (*cf.* the end of Section 4.4).

The expressions for the physical electromagnetic field on the earth's surface and for the physical potential difference between P_1 and P_2 are equal to the real parts of formulae (4.184)–(4.190).

As in Chapter 3, the assumed conductivity σ_0 of the air may have an arbitrarily small (but positive) value. Thus equations (4.184)–(4.191) are valid when the properties of the upper half-space are arbitrarily close to those of free space or ideal air. Owing to the »continuity of physics» it is obvious that the limit of formulae (4.184)–(4.191) with σ_0 approaching zero gives $\bar{E}_M$, $\bar{B}_M$ and $U_{P_1P_2}$ for free space, and evidently this limit is simply obtained by setting $\sigma_0 = 0$.

The ratios of the integrands of formulae (4.184)–(4.189) can be expressed in terms of q, ω and the electromagnetic parameters of the air and the earth. These ratios are independent of J and h, but contrary to Section 3.4 thus depend on all the electromagnetic properties of the air, not only on μ_0 (*cf.* the impedances (4.133) and (4.134)).

4.6. Induction in a homogeneous earth

Let us now assume that the earth is homogeneous with parameters σ, ϵ, μ, k, η_1 and κ. The subscript 1 is preserved in η_1, since η is associated with the air (formulae (4.8) and (4.9)). As mentioned in Section 4.5 the coefficients $c_g(b)$ and $m_f(b)$ are zero and $c_f(b)$ and $m_g(b)$ equal $-\kappa$. Hence according to equations (4.181)–(4.190) the total electromagnetic field on the surface of a homogeneous earth and the potential difference between the points $P_1 = (x_1, y_1, 0)$ and $P_2 = (x_2, y_2, 0)$ have the following expressions:

$$E_{Mx}(x,y,t) = -\frac{\omega\mu_0 q\eta^2\eta_1^2 J e^{i(\omega t - qy)}}{\pi}\int_{-\infty}^{\infty} db \left[b\left(\kappa_0 + \frac{\mu}{\mu_0}\kappa\right) e^{-\kappa_0 h}\cdot \sin bx \middle/ \left(q^2b^2(k^2-k_0^2)^2 - \left(\eta_1^2\kappa_0 + \frac{\mu}{\mu_0}\eta^2\kappa\right)\left(k_0^2\eta_1^2\kappa_0 + \frac{\mu_0}{\mu}k^2\eta^2\kappa\right)\right)\right], \tag{4.192}$$

$$E_{My}(x,y,t) = \frac{i\omega\mu_0 \eta^2\eta_1^2 J e^{i(\omega t - qy)}}{\pi}\int_{-\infty}^{\infty} db \left[\left(\eta_1^2\kappa_0 + \frac{\mu}{\mu_0}\eta^2\kappa\right) e^{-\kappa_0 h}\cdot \cos bx \middle/ \left(q^2b^2(k^2-k_0^2)^2 - \left(\eta_1^2\kappa_0 + \frac{\mu}{\mu_0}\eta^2\kappa\right)\left(k_0^2\eta_1^2\kappa_0 + \frac{\mu_0}{\mu}k^2\eta^2\kappa\right)\right)\right], \tag{4.193}$$

$$E_{Mz}(x,y,t) = -\frac{\omega\mu_0 q k^2\eta^2 J e^{i(\omega t - qy)}}{\pi k_0^2}\int_{-\infty}^{\infty} db\Bigg[\Big(b^2(k^2-k_0^2)+ \tag{4.194}$$

$$+\frac{\mu_0}{\mu}\kappa\Big(\eta_1^2\kappa_0+\frac{\mu}{\mu_0}\eta^2\kappa\Big)\Big)e^{-\kappa_0 h}\cdot\cos bx\Big/\Big(q^2b^2(k^2-k_0^2)^2+$$

$$-\Big(\eta_1^2\kappa_0+\frac{\mu}{\mu_0}\eta^2\kappa\Big)\Big(k_0^2\eta_1^2\kappa_0+\frac{\mu_0}{\mu}k^2\eta^2\kappa\Big)\Big)\Bigg],$$

$$B_{Mx}(x,y,t) = -\frac{\mu_0\eta^2 J e^{i(\omega t - qy)}}{\pi}\int_0^{\infty} db\Bigg[\Big(q^2b^2(k^2-k_0^2)+ \tag{4.195}$$

$$+\frac{\mu_0}{\mu}k^2\kappa\Big(\eta_1^2\kappa_0+\frac{\mu}{\mu_0}\eta^2\kappa\Big)\Big)e^{-\kappa_0 h}\cdot\cos bx\Big/\Big(q^2b^2(k^2-k_0^2)^2+$$

$$-\Big(\eta_1^2\kappa_0+\frac{\mu}{\mu_0}\eta^2\kappa\Big)\Big(k_0^2\eta_1^2\kappa_0+\frac{\mu_0}{\mu}k^2\eta^2\kappa\Big)\Big)\Bigg],$$

$$B_{My}(x,y,t) = -\frac{i\mu_0 q(k^2-k_0^2)\eta^2\eta_1^2 J e^{i(\omega t - qy)}}{\pi}\int_0^{\infty} db\Bigg[be^{-\kappa_0 b}\cdot\sin bx\Big/\Big(q^2b^2(k^2+$$

$$-k_0^2)^2-\Big(\eta_1^2\kappa_0+\frac{\mu}{\mu_0}\eta^2\kappa\Big)\Big(k_0^2\eta_1^2\kappa_0+\frac{\mu_0}{\mu}k^2\eta^2\kappa\Big)\Big)\Bigg], \tag{4.196}$$

$$B_{Mz}(x,y,t) = \frac{\mu_0\eta^2\eta_1^2 J e^{i(\omega t - qy)}}{\pi}\int_0^{\infty} db\Bigg[b\Big(k^2\kappa_0+\frac{\mu}{\mu_0}k_0^2\kappa\Big)e^{-\kappa_0 h}\cdot\sin bx\Big/\Big(q^2b^2(k^2+$$

$$-k_0^2)^2-\Big(\eta_1^2\kappa_0+\frac{\mu}{\mu_0}\eta^2\kappa\Big)\Big(k_0^2\eta_1^2\kappa_0+\frac{\mu_0}{\mu}k^2\eta^2\kappa\Big)\Big)\Bigg] \tag{4.197}$$

and

$$U_{P_1P_2}(t) = \frac{\omega\mu_0\eta^2\eta_1^2 J e^{i\omega t}}{2\pi}\int_{-\infty}^{\infty} db\Bigg[\big(e^{i(bx_2-qy_2)}-e^{i(bx_1-qy_1)}\big)\Big((x_2-x_1)qb\Big(\kappa_0+$$

$$+\frac{\mu}{\mu_0}\kappa\Big)+(y_2-y_1)\Big(\eta_1^2\kappa_0+\frac{\mu}{\mu_0}\eta^2\kappa\Big)\Big)e^{-\kappa_0 h}\Big/\Big((b(x_2-x_1)-q(y_2+$$

$$-y_1))\Big(q^2b^2(k^2-k_0^2)^2-\Big(\eta_1^2\kappa_0+\frac{\mu}{\mu_0}\eta^2\kappa\Big)\Big(k_0^2\eta_1^2\kappa_0+\frac{\mu_0}{\mu}k^2\eta^2\kappa\Big)\Big)\Big)\Bigg]. \tag{4.198}$$

Formula (A.29) and the clear oddness and evenness of the integrands have been utilized in the derivation of equations (4.192)–(4.197). $E_{Mx}(x, y, t)$, $B_{My}(x, y, t)$ and $B_{Mz}(x, y, t)$ are odd as functions of x, while $E_{My}(x, y, t)$, $E_{Mz}(x, y, t)$ and $B_{Mx}(x, y, t)$ are even with respect to x. Equations (4.192)–(4.198) are comparable to formulae (3.66)–(3.69), which are also associated with a homogeneous earth but without any y-dependence of the source. For $q = 0$ formulae (4.193), (4.195) and (4.197) reduce to equations (3.66)–(3.68). The latter must be a little modified, if the permeability of the earth differs from the free space value. With $q = 0$ equations (4.192), (4.194) and (4.196) yield zero, as expected, and formula (4.198), of course, gives equation (3.69).

In order to somewhat elucidate the influence of the harmonic space dependence of the source, *i.e.* the effect of q, the component $E_{My}(x, y, t)$ will be considered more closely. The conductivity σ_0 is almost zero and for simplicity let us assume that μ is equal to μ_0 and that the inequalities $\sigma \gg \omega\epsilon, \omega\epsilon_0$ are satisfied (*cf.* Section 2.3). Then $|k|$ is much larger than $|k_0|$. Assume further that q is much larger than $|k|$. Then the following approximations are valid: $\eta \approx \eta_1 \approx -iq$, $\kappa_0 \approx \kappa \approx \sqrt{b^2 + q^2}$. To further simplify the discussion let us set x equal to zero and so consider E_{My} under the primary source. The numerator and the denominator of the integrand of formula (4.193) can now be approximated as follows:

$$\left(\eta_1^2\kappa_0 + \frac{\mu}{\mu_0}\eta^2\kappa\right)e^{-\kappa_0 h} \approx -2q^2\sqrt{b^2+q^2}\, e^{-\sqrt{b^2+q^2}\,h} \tag{4.199}$$

and

$$\begin{aligned} q^2b^2(k^2 - k_0^2)^2 &- \left(\eta_1^2\kappa_0 + \frac{\mu}{\mu_0}\eta^2\kappa\right)\left(k_0^2\eta_1^2\kappa_0 + \frac{\mu_0}{\mu}k^2\eta^2\kappa\right) \\ &\approx q^2b^2k^4 - 2q^2b^2k^2k_0^2 + q^2b^2k_0^4 - 2q^4(b^2+q^2)(k_0^2+k^2) \\ &= q^2b^2(k^4 - 2k^2k_0^2 + k_0^4 - 2q^2k_0^2 - 2q^2k^2) - 2q^6(k_0^2+k^2) \\ &\approx -2q^4k^2(b^2+q^2)\,. \end{aligned} \tag{4.200}$$

By substituting these approximations into formula (4.193) $E_{My}(x = 0, y, t)$ has the expression

$$E_{My}(x=0, y, t) \approx \frac{i\omega\mu_0 q^2 J e^{i(\omega t - qy)}}{\pi k^2}\int_0^\infty \frac{e^{-\sqrt{b^2+q^2}\,h}}{\sqrt{b^2+q^2}}\,db\,. \tag{4.201}$$

The integral appearing in equation (4.201) can be calculated using formula (A.74) and so

$$E_{My}(x=0,y,t) \approx \frac{\omega\mu_0 q^2 J e^{i(\omega t - qy)}}{2k^2} \; H_0^{(2)}(-iqh) \,. \tag{4.202}$$

Let us assume that q is also much larger than the inverse of h. Then the asymptotic expression of the Hankel function (A.59) may be substituted into equation (4.202) and the result is

$$E_{My}(x=0,y,t) \approx \frac{i\omega\mu_0 J e^{i(\omega t - qy)}}{k^2 h^2} \, (2\pi)^{-1/2}(qh)^{3/2} e^{-qh} \,. \tag{4.203}$$

The function $(qh)^{3/2}e^{-qh}$ decreases as q increases when q is larger than $3/2h$. According to the assumption $q \gg 1/h$ made above q really lies in that range. Formula (4.203) therefore indicates that an increase of q, with the other parameters kept constant, decreases the amplitude of the electric field component E_{My}, which oscillates harmonically with time, at every point on the earth's surface under the primary source.

For comparison let us discuss the case $q = 0$, $\sigma_0 \approx 0$, $\mu = \mu_0$ and $\sigma \gg \omega\epsilon, \omega\epsilon_0$. Then the parallel component of the electric field at the earth's surface under the primary source is given by formula (3.66) with $x = 0$. As $|k_0|$ is much smaller than $|k|$ let us use the approximative equation (3.71). Due to the term e^{-bh} the integral of formula (3.71) gets its largest contribution at small values of b. Then obviously the denominator $b + \kappa = b + \sqrt{b^2 - k^2}$ may be replaced in the integration by ik. The result is

$$E_{My}(x=0,t) \approx -\frac{\omega\mu_0 J e^{i\omega t}}{\pi k h} \,. \tag{4.204}$$

This formula is also obtained from equation (3.74) with $x = 0$ and with the utilization of the asymptotic expansion of $Y_1(ikh) - H_1(ikh)$ (A.79).

Equations (4.203) and (4.204) show that the presence of a large longitudinal propagation constant q diminishes the amplitude of E_{My} by a coefficient $(\pi/2)^{1/2}(qh)^{3/2}e^{-qh}/(|k|h)$ compared to the case of no longitudinal propagation. A phase shift also occurs. Assume for example that the parameters have the following values: $\omega = 1\ s^{-1}$, $\sigma = 10^{-3}\Omega^{-1}m^{-1}$ and $h = 10^5$ m (*cf.* SARAOJA, 1946, p. 122, ALBERTSON and VAN BAELEN, 1970, KAUFMAN and KELLER, 1981, pp. 3 and 24). Then the approximate values of $|k|$ and $|k_0|$ are $3.5 \cdot 10^{-5}\ m^{-1}$ and $3.3 \cdot 10^{-9}\ m^{-1}$. For the assumption $q \gg |k|$ to be valid, q must be of the order of $10^{-3}\ m^{-1}$. Then e^{-qh} makes the ratio of the amplitudes of the electric field components expressed by formulae (4.203) and (4.204) vanishingly small.

In the case $\mu = \mu_0$, $x = 0$, $|k| \gg q \gg |k_0|$ and $q \ll 1/h$ the approximated result for $E_{My}(x=0, y, t)$ does not differ from formula (4.204) corresponding to the case $q = 0$ (excluding the factor e^{-iqy}).

With the numerical values $\omega = 100\ \mathrm{s}^{-1}$, $\sigma = 10^{-3}\Omega^{-1}\mathrm{m}^{-1}$, $h = 10^5$ m and $q = 3 \cdot 10^{-5}\mathrm{m}^{-1}$ (and $\mu = \mu_0$, $x = 0$), which satisfy the condition $|k| \gg q \gg |k_0|$, but not $q \ll 1/h$, the rough result is that the amplitude of E_{My} is of the order of one tenth of the value connected with no y-dependence, and the only phase difference is due to the term e^{-iqy}. The value of ω is high from the point of view of geomagnetic variations (KELLER and FRISCHKNECHT, 1970, p. 203).

Although the approximative considerations above cannot be regarded as mathematically definite, they seem to indicate that longitudinal propagation in the primary source current tends to decrease the parallel electric field component at the earth's surface under the source. As mentioned in Section 3.3 after the approximative formulae (3.71)–(3.73), an approximation can safely be accepted only when the exact and the approximative expressions are directly compared. This would, however, be very complicated in the present case and is therefore neglected.

The purpose of the above discussion has been to evaluate the influence of q on the electromagnetic field. Geophysically reasonable values for q have not been looked for. A reasonable value of q might in practice be some $10^{-6}...10^{-5}\ \mathrm{m}^{-1}$, which corresponds to a longitudinal wavelength ($2\pi/q$) of hundreds to thousands of kilometers.

The conclusions, of course, only concern the electric field component $E_{My}(x = 0, y, t)$ discussed above, which is parallel to the primary source.

4.7. The case of no longitudinal space dependence of the primary source

Let us now consider the case where the longitudinal space dependence of the source current and charge vanishes, *i.e.* $q = 0$. (The earth again has the most general properties valid in equations (4.184)–(4.190).) The assumption $q = 0$, which has already been referred to in equations (4.137) and (4.138) and in the discussion of formulae (4.192)–(4.198), yields the results obtained in Chapter 3. In the latter, however, no horizontal variation in the electromagnetic parameters of the earth was included.

Maxwell's equations (2.93)–(2.96) determining the induced field within the earth resolve into two sets of equations: one containing only E_y, H_x and H_z, the other containing only E_x, E_z and H_y. Hence the functions $f(x, z)$ and $g(x, z)$ appearing in the expressions of $E_y(x, y, z, t)$ and $H_y(x, y, z, t)$ in equations (4.153) and (4.154) are separate, from which it can be concluded that the coefficients $c_g(b)$ and $m_f(b)$ are zero (see formulae (4.165) and (4.166)). The quantities η and

η_1 are equal to k_0 and k_1, respectively. Thus formulae (4.184)–(4.190) for $q = 0$ can be written as

$$E_{Mx}(x, t) = 0 \,, \tag{4.205}$$

$$E_{My}(x, t) = -\frac{i\omega\mu_0 J e^{i\omega t}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\kappa_0 h}}{\kappa_0 - \frac{\mu_0}{\mu_1} c_f} e^{ibx} db \,, \tag{4.206}$$

$$E_{Mz}(x, t) = 0 \,, \tag{4.207}$$

$$B_{Mx}(x, t) = \frac{\mu_0 J e^{i\omega t}}{2\pi} \int_{-\infty}^{\infty} \frac{c_f e^{-\kappa_0 h}}{c_f - \frac{\mu_1}{\mu_0} \kappa_0} e^{ibx} db \,, \tag{4.208}$$

$$B_{My}(x, t) = 0 \,, \tag{4.209}$$

$$B_{Mz}(x, t) = \frac{i\mu_0 J e^{i\omega t}}{2\pi} \int_{-\infty}^{\infty} \frac{b e^{-\kappa_0 h}}{\kappa_0 - \frac{\mu_0}{\mu_1} c_f} e^{ibx} db \tag{4.210}$$

and

$$U_{P_1 P_2}(t) = -\frac{\omega\mu_0 (y_2 - y_1) J e^{i\omega t}}{2\pi (x_2 - x_1)} \int_{-\infty}^{\infty} \frac{e^{-\kappa_0 h}}{b\left(\kappa_0 - \frac{\mu_0}{\mu_1} c_f\right)} (e^{ibx_2} - e^{ibx_1}) db. \tag{4.211}$$

The quantity κ_0 is given by formulae (4.62) and (4.63) with $q = 0$ and is thus identical with the quantity κ_0 of Chapter 3 (see formulae (3.16) and (3.17)). Equations (4.205), (4.207) and (4.209) agree with the conclusions made in Chapter 3 that the electromagnetic field does not have the components E_x, E_z and B_y.

Let us define a surface impedance $Z(b)$ as

$$Z(b) = -\frac{E_y(b, x, y, z = 0, t)}{H_x(b, x, y, z = 0, t)} \tag{4.212}$$

where $E_y(b, x, y, z = 0, t)$ and $H_x(b, x, y, z = 0, t)$ are the integrands in formulae (4.155) and (4.159) with $z = 0$, *i.e.*

$$E_y(b, x, y, z = 0, t) = f(b) e^{i(\omega t - qy + bx)} \tag{4.213}$$

and

$$H_y(b, x, y, z = 0, t) = \left(-\frac{(\sigma_1 + i\omega\epsilon_1)}{\eta_1^2} f'(b) + qbg(b) \right) e^{i(\omega t - qy + bx)} \tag{4.214}$$

(*cf.* equation (4.133) associated with the horizontally layered earth model). In the present case $q = 0$, $Z(b)$ has the expression

$$Z(b) = -i\omega\mu_1 \frac{f(b)}{f'(b)} = -\frac{i\omega\mu_1}{c_f(b)} \tag{4.215}$$

(*cf.* equation (4.137)). The impedance $Z(b)$ is thus only a function of the properties of the earth and is obtained by assuming a harmonic time dependence and that q is zero. The latter equality in formula (4.215) was obtained from equation (4.165) using the fact that $c_g(b)$ is zero.

Substitution of $c_f(b) = -i\mu_1\omega/Z(b)$ in equations (4.206), (4.208), (4.210) and (4.211) yields

$$E_{My}(x, t) = -\frac{i\omega\mu_0 J e^{i\omega t}}{2\pi} \int_{-\infty}^{\infty} \frac{Z(b) e^{-\kappa_0 h}}{\kappa_0 Z(b) + i\omega\mu_0} e^{ibx} db\,, \tag{4.216}$$

$$B_{Mx}(x, t) = \frac{i\omega\mu_0^2 J e^{i\omega t}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\kappa_0 h}}{\kappa_0 Z(b) + i\omega\mu_0} e^{ibx} db\,, \tag{4.217}$$

$$B_{Mz}(x, t) = \frac{i\mu_0 J e^{i\omega t}}{2\pi} \int_{-\infty}^{\infty} \frac{bZ(b) e^{-\kappa_0 h}}{\kappa_0 Z(b) + i\omega\mu_0} e^{ibx} db \tag{4.218}$$

and

$$U_{P_1P_2}(t) = -\frac{\omega\mu_0(y_2 - y_1) J e^{i\omega t}}{2\pi(x_2 - x_1)} \int_{-\infty}^{\infty} \frac{Z(b) e^{-\kappa_0 h}}{b(\kappa_0 Z(b) + i\omega\mu_0)} (e^{ibx_2} - e^{ibx_1}) db\,. \tag{4.219}$$

If $Z(b)$ were assumed to be even with respect to b, it would be possible to make formulae (4.216)–(4.219) identical to equations (3.60)–(3.63). As seen from formula (4.215), the evenness of $Z(b)$ is equivalent with the evenness of $c_f(b)$. Formulae (3.60)–(3.63) were shown valid in the case of vertical variation only in the electromagnetic parameters of the earth. Let us therefore reject the x-dependence of the parameters also now. Equation (4.151) determining the component E_y in the earth then reduces to formula (3.84). The treatments in Sections 3.4 and 4.5 show that the function $f(b, z)$ can now be equated in each continuous region with the function $D_j(b)g_{jb}^-(z) + G_j(b)g_{jb}^+(z) = G_1(b)(\gamma_j(b)g_{jb}^-(z) + \xi_j(b)g_{jb}^+(z))$. Since $\gamma_j(b)$, $\xi_j(b)$, $g_{jb}^-(z)$ and $g_{jb}^+(z)$ are even with respect to b, the ratios

$\frac{f'(b)}{f(b)} = \left[\frac{1}{f(b,z)} \frac{\partial f(b,z)}{\partial z}\right]_{z=0}$ and $\frac{f'(-b)}{f(-b)} = \left[\frac{1}{f(-b,z)} \frac{\partial f(-b,z)}{\partial z}\right]_{z=0}$ are equal, which according to formula (4.165), where $c_g(b) = 0$, implies that $c_f(b)$ is even with respect to b. Thus equations (4.216)–(4.219) can be expressed as formulae (3.60)–(3.63), as is expected.

It is clear that $E_y(x, z, t)$ in the earth is even with respect to x if the parameters of the earth are even functions in x, not necessarily independent of x. Then $f(b, z) = f(-b, z)$ (see equations (4.153) and (4.155)) and further $c_f(b) = c_f(-b)$. In such a more general situation formulae (3.60)–(3.63) are thus also obtained. The same assumption of the evenness in x of the properties of the earth would probably permit formulae (4.184)–(4.189), which are associated with the case of any non-negative value of q, to be written as b-integrals from zero to infinity such that the integrals involve $\sin bx$ or $\cos bx$ instead of e^{ibx} (*cf.* the special case of equations (4.192)–(4.197)).

5. *Discussion and concluding remarks*

This paper deals with three theoretical models describing the primary field of electromagnetic induction in the earth:

1. A harmonic plane wave propagating vertically downwards, corresponding to an infinite horizontal current sheet above the earth's surface as the primary source (Chapter 2).
2. The field caused by an infinitely long horizontal straight line current oscillating harmonically in time, and situated above the earth's surface (Chapter 3).
3. The field caused by a similar line current which, in addition, has a longitudinal harmonic space dependence, implying the existence of charges on the line (Chapter 4).

In all these cases the earth is assumed to be an infinite half-space with a plane boundary. Hence the treatments can be used in local induction studies involving areas in the order of hundreds of kilometres. On a global scale the sphericity of the earth has to be taken into account. Obviously, too, in the half-space model the electromagnetic field should not penetrate deeper in the earth than the dimensions of the area investigated (*cf.* FRAZER, 1974, pp. 402–403). In a homogeneous earth the skin depth is simply a measure of this depth of penetration. KAUFMAN and KELLER, 1981, pp. 157–174, say that the sphericity of the earth becomes significant only when the skin depth is not small compared to the radius of the earth. The effect of the earth's curvature is also discussed by WAIT, 1962, pp. 532–539.

The upper half-space, the air, is initially assumed to behave as electromagnetically free space, which is a good approximation to reality. For mathematical reasons a slight conductivity is assumed for the air in Chapters 3 and 4. In reality, the conductivity of the air is in the order of $10^{-14}\Omega^{-1}m^{-1}$ near the earth's surface (ISRAËL, 1971, pp. 95 and 249).

In this paper the electromagnetic properties of the earth are mainly assumed to be laterally constant and piecewise constant in the vertical direction, *i.e.* the earth is horizontally layered. Such a model is an idealization of the real situation. This work, however, also contains extensions to arbitrary vertical variations of the electromagnetic parameters of the earth. In Chapter 4, even variations in one horizontal direction are included. These extensions bring the earth models closer to the true situation, though their inclusion is merely formal, and they contain functions whose determination from their differential equations is probably complicated in actual practice.

In reality the propagation constant of the earth is much larger than that of the air. Thus a plane wave incident with any real angle to the earth's surface is refracted in the earth approximately as a plane wave propagating vertically downwards, as implied by Snell's law. So the results obtained with the first model for the primary field are evidently applicable to any direction of propagation of the primary plane wave, provided the angle of incidence is real (see CAGNIARD, 1953, pp. 613–614, WAIT, 1954, pp. 282, 286 and 287, and WAIT, 1962, p. 526). Oblique angles may, however, also imply a vertical electric field component associated with surface charge at the earth's surface. The conclusion on the validity of the first model can be expressed even more generally by stating that the results obtained in Chapter 2 are also applicable to complex angles of incidence, provided the wave number describing the horizontal space dependence (*i.e.* the horizontal propagation constant) is small compared to the propagation constant of the earth. The inverse of this horizontal wave number equals a typical changing distance of the field in the horizontal direction, and the inverse of the propagation constant of the earth expresses the skin depth. So in other words, the less the field varies over a horizontal distance equal to the skin depth of the earth the more usable are the results of Chapter 2 (WAIT, 1954, p. 282, and 1962, pp. 526–532). Clearly, the farther away the primary source, the less the field varies in space. (The above statements of the propagation constant and of the skin depth are clear in the case of a homogeneous earth. If these quantities are not constant in the earth, the statements should evidently hold for all their relevant values.) On the applicability of Chapter 2, see also SRIVASTAVA, 1965, QUON *et al.*, 1979, JONES, 1980, and KAUFMAN and KELLER, 1981, pp. 113–155.

In local induction studies, the primary source of the infinitely long straight line current provides the simplest model for auroral and equatorial electrojet currents flowing geomagnetically east to west in the ionosphere approximately 100 km above the earth's surface. In practice, these electrojets have non-zero dimensions in the transverse directions. Such models are implied by Chapter 3, because they can be constructed as superpositions of line currents and because the equations used in the calculations are linear (see below).

The case of an electrojet approximated by an infinitely long horizontal current sheet whose width is finite lies somewhere between the first and the second models. According to ALBERTSON and VAN BAELEN, 1970, the plane wave model sets the upper limit on the electric field observed on the earth's surface, while the lower limit can be obtained using a line current model (Fig. 2). HERMANCE and PELTIER, 1970, point out that fields caused by horizontal current sheets can be approximated by those of line currents situated at greater heights.

The addition to the line current of a longitudinal sinusoidal space dependence generalizes the model because it permits variations along the electrojet. The first effect of this addition to be found on comparing the results of Chapters 3 and 4 is that the electric field also has components perpendicular to the source, and the magnetic field has a parallel component. The preliminary evaluation carried out in Section 4.6 indicates that the space dependence of the primary source tends to diminish the parallel electric field component on the earth's surface. To obtain more information on the influence of the longitudinal space dependence, however, exact numerical calculations would have to be made. Chapter 4, like Chapter 3, implies models in which the primary current, harmonic in time and harmonic with respect to the longitudinal space-coordinate, has non-zero transverse dimensions.

The source currents of the second and third models were mainly assumed to flow in the east-west direction, *i.e.* approximately the direction of the electrojets. By rotating the coordinate system, any horizontal direction desired for the currents can, however, be obtained.

Only harmonic time-dependence is considered in this paper, except in Section 2.5. But since any »sufficiently» regular function of time can be expressed as a Fourier integral of harmonic components (Section A.4), the discussions in this work are also applicable to arbitrary time variations of the primary sources. But a superposition of this kind, as used in Section 2.5, is possible only if the media are linear, as they are in this paper (see Section B.4). In fact, the linearity of the media also makes it permissible to omit the main geomagnetic field when dealing with the electromagnetic induction in the earth (see Chapter 1). If non-linear media were present, it would not be possible to use complex quantities (*cf.* Section A.1). Further, the assumption made in Chapters 2, 3 and 4 that $e^{i\omega t}$ is the only time dependence would be incorrect. It also presumes that the electromagnetic parameters of the media are independent of time.

With fully analogous comments, Chapter 4 can be regarded as relating to a space-Fourier component of an arbitrary longitudinal space dependence of the primary source.

The main result of this work is the development of a theory of electromagnetic induction in the earth caused by an infinitely long, horizontal line current describing an electrojet, which oscillates harmonically in time and also has a longitudinal harmonic space dependence. It is demonstrated that the electromagnetic field on the earth's surface can be calculated theoretically, as a function of the primary source and of the properties of the earth, from Maxwell's equations and from electromagnetic boundary conditions if the earth is horizontally layered. Formal extensions to arbitrary vertical variations of the electromagnetic parameters of the earth and to lateral variations perpendicular to the current are also made. In the mathematical calculation, however, it proved necessary to assume that the lateral dependence vanishes at the earth's surface. To start with a longitudinal exponential attenuation of the primary current was also included in the model. But it had to be rejected, because, otherwise, there would have been no finite primary field.

The final formulae for the field on the earth's surface are complicated integrals over a horizontal wave number. In future, when the results of this work are put into practice, the formulae should be simplified by approximations, and numerical integration will be needed. As regards further development of the theory, the model of the primary source could be improved, say, by adding vertical currents starting upwards from the present current. Such currents would affect the accumulated charge.

Formulations of the plane wave model and of the time-harmonic line current model without harmonic space dependence are included in this work to provide complete and non-approximative treatments of these subjects. Comments are presented on previous publications dealing with these models. Nevertheless, the main results derived by earlier authors are good approximations of rigorous formulae in the case of electromagnetic induction in the earth. The treatment of the time-harmonic line current model is also an introduction to the third model, whose method of calculation is the same.

In the case of the time-harmonic line current, the electromagnetic field on the earth's surface is expressed in this paper as a function of the primary current and of the properties of the earth. In the plane-wave case only the relationship between the electric and the magnetic field is expressed. This relationship is a function of the properties of the earth. If the phenomenon is caused by the infinite horizontal current sheet assumed initially, in which case the formulae are exact, the electromagnetic field on the earth's surface can be expressed as a function of the primary plane wave using the treatment in Chapter 2 directly. The primary wave throws light on the properties of the current sheet.

Since the calculations in this paper are based rigorously on classical electromagnetic theory, the results are applicable to any values of the parameters, and not only to those reasonable for electromagnetic induction in the earth, which permit displacement currents to be neglected. So this work may also be useful in solving other kinds of electromagnetic problems.

Appendix A. Mathematical subjects and formulae

A.1. Complex and physical quantities

In the mathematical treatment of physical problems it is often convenient to use complex quantities. The actual physical quantities are, of course, real, and so a convention can be made that the real part of a complex expression is the physical quantity. Normally the time-dependence of the complex expressions is harmonic, *i.e.* $e^{i\omega t} = \cos\omega t + i\sin\omega t$, where ω is the real angular frequency (or its opposite). Let $G(r)e^{i\omega t}$ be some complex field. The corresponding physical field is then

$$G_{phys}(\bar{r},t) = Re[G(\bar{r})e^{i\omega t}] = \frac{1}{2}\,(G(\bar{r})e^{i\omega t} + G^{*}(\bar{r})e^{-i\omega t}). \tag{A.1}$$

The asterisk denotes the complex conjugate. (See *e.g.* STRATTON, 1941, pp. 135–136, JONES, 1964, p. 53, PANOFSKY and PHILLIPS, 1964, p. 190).

The use of complex quantities is correct, if all operations involved are linear. In the manipulation of products of physical quantities we must, however, be careful, because the real part of the product of two complex quantities is not equal to the product of their real parts (see Section B. 9).

A physical quantity is not allowed to grow to infinity, but we may ask, whether finiteness can be required of complex quantities, because their imaginary parts need not behave physically. It can, however, be shown that in the case of a harmonic time-dependence with $\omega \neq 0$, the complex quantities must also remain finite. Similarly the space continuity of a physical quantity implies the same property of the corresponding complex quantity, if the time-dependence is harmonic with $\omega \neq 0$.

In this work harmonic electromagnetic quantities ($\omega \neq 0$) are discussed, and it can be shown that the treatment of complex quantities in Maxwell's equations and boundary conditions is then equivalent to the treatment of the corresponding physical quantities (see Appendix B).

A.2. Convention of the arguments of complex square roots

Because square roots are double-valued, a clear definition of the range of their arguments is always necessary. In this work, unless otherwise indicated, we assume that the arguments of complex numbers z are chosen in the range $-\pi < \arg z \leq \pi$ and that the argument of $\sqrt{z} = z^{1/2}$ is equal to $(1/2)\arg z$. Hence square roots in this work normally lie in the half-plane $-\pi/2 < \arg\sqrt{z} \leqq \pi/2$. More generally, the choice of the argument of z is also important in formulae $\log z = \log|z| + i \arg z$ and $z^{\alpha} = e^{\alpha \log z}$ where α is any real or complex number.

A.3. Dirac delta function

The one-dimensional Dirac delta function $\delta(t)$, which is actually not a function but a distribution, can be defined as

$$\delta(t) = \lim_{\Delta \to 0+} \begin{cases} 0, & t \leqslant -\frac{\Delta}{2} \\ \frac{1}{\Delta}, & -\frac{\Delta}{2} < t < \frac{\Delta}{2} \\ 0, & t \geqslant \frac{\Delta}{2} \end{cases} \tag{A.2}$$

(MORSE and FESHBACH, 1953, p. 122). In other words

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ +\infty, & t = 0 \end{cases} . \tag{A.3}$$

The most useful property of the delta function is the following integral relation

$$\int_a^b f(t')\delta(t-t')\,dt' = \begin{cases} f(t), & a < t < b \\ \frac{f(t)}{2}, & t = a \text{ or } t = b \\ 0, & t < a \text{ or } t > b \end{cases} \tag{A.4}$$

where $f(t)$ is an arbitrary function of a real argument t, a and b are real, possibly infinite, and $a < b$ (*cf.* JONES, 1964, pp. 35–36).

The delta function is the derivative of the so-called step function $\theta(t)$ defined by

$$\theta(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases} \tag{A.5}$$

(MORSE and FESHBACH, 1953, p. 123). A formal equation

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\alpha t}\, d\alpha, \tag{A.6}$$

which actually is the inverse Fourier transform of $1/\sqrt{2\pi}$ (Section A.4), is also very useful. Because $\delta(t)$ is even with respect to t, a minus sign can equally well be put into the exponent of $e^{i\alpha t}$.

A.4. Fourier transform

The one-dimensional Fourier transform $f(\omega)$ of a function $f(t)$ of a real argument t is

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \tag{A.7}$$

The function $f(t)$ is expressible as the inverse Fourier transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} d\omega \tag{A.8}$$

(see *e.g.* WIENER, 1933, p. 3, MORSE and FESHBACH, 1953, p. 453). It is natural that the function $f(t)$ has to be sufficiently regular in order that its Fourier transform exists. In this work, however, it is not necessary to deal with the mathematical theory of Fourier transforms. So we just refer to *e.g.* WIENER, 1933, ARSAC, 1966, and MORSE and FESHBACH, 1953, pp. 453–471.

Formula (A.7) indicates that $f(\omega)$ does not change if the values of $f(t)$ undergo finite changes in a set of measure zero, *e.g.* at separate points. Similarly changes of $f(\omega)$ in a set of measure zero do not affect the value of $f(t)$ given by equation (A.8). Thus $f(t)$ of equation (A.7) may differ from $f(t)$ of (A.8) in a set of measure zero, and the same is also true for $f(\omega)$.

Let us denote the derivative $\frac{df(t)}{dt}$ by $g(t)$. Application of equation (A.7) then using a partial integration implies that

$$g(\omega) = i\omega f(\omega) \tag{A.9}$$

provided that $f(t)$ is zero for $t = \pm\infty$.

Substitution of equation (A.7) in (A.8) gives

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dt' f(t') e^{i\omega(t-t')}, \tag{A.10}$$

which is known as the Fourier integral theorem. A similar integral formula is also satisfied by $f(\omega)$. If the order of integration is formally changed in equation (A.10) and the equation is then compared with formula (A.4) with $a = -\infty$ and $b = +\infty$, the validity of equation (A.6) can be concluded.

On the other hand, according to what was said above the functions $f(t)$ on different sides of formula (A.10) may have different values in a set of measure zero. But if formula (A.6) is considered valid, these functions apparently become equal at every point. Similarly, if the equation

$$\int_{-\infty}^{\infty} f(\omega)e^{i\omega t}d\omega = \int_{-\infty}^{\infty} g(\omega)e^{i\omega t}d\omega, \tag{A.11}$$

satisfied for every value t, is multiplied by $\frac{1}{2\pi}e^{-i\omega' t}$ and integrated first with respect to t from $-\infty$ to $+\infty$, the formal use of equations (A.6) and (A.4) yields

$$f(\omega) = g(\omega) \tag{A.12}$$

for every ω. But equation (A.11) is valid although $f(\omega)$ and $g(\omega)$ differ in a set of measure zero. If integrals such as those appearing in equation (A.11) are considered this phenomenon is, however, of no importance, since changes in the values of $f(\omega)$ and $g(\omega)$ in a set of measure zero have no influence. Therefore equation (A.6) can be used in the present work.

Let us denote the Fourier transforms of $f(t)$ and $g(t)$ by $F(\omega)$ and $G(\omega)$, respectively. The use of the definitions (A.7) and (A.8) with the interchange of the order of integration then shows that

$$\int_{-\infty}^{\infty} F(\omega)\, G(\omega)e^{i\omega t}d\omega = \int_{-\infty}^{\infty} f(t-u)g(u)\,du = \int_{-\infty}^{\infty} f(u)g(t-u)\,du. \tag{A.13}$$

This formula is the so-called convolution or Faltung theorem (MORSE and FESHBACH, 1953, pp. 464–465).

Let us assume that $f(t)$ is a real function (*i.e.* physical). Formula (A.7) then implies that

$$f(\omega) = f^{\star}(-\omega). \tag{A.14}$$

From equations (A.8) and (A.14) we obtain

$$f(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} Re[f(\omega)e^{i\omega t}]\,d\omega = Re\left[\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(\omega)e^{i\omega t}\,d\omega\right]. \tag{A.15}$$

A.5. Asymptotic expressions

Let $f(z)$ and $\varphi(z)$ be two complex functions of a complex variable z. The series $\sum_{p=0}^{\infty} a_p/z^p$ where the quantities a_p are complex constants is said to represent $f(z)/\varphi(z)$ asymptotically, if

$$\lim_{|z|\to\infty} \left[z^n \left(\frac{f(z)}{\varphi(z)} - \sum_{p=0}^{n} \frac{a_p}{z^p} \right) \right] = 0 \tag{A.16}$$

for every non-negative integer value of n (MORSE and FESHBACH, 1953, p. 434). We then write

$$f(z) \simeq \varphi(z) \sum_{p=0}^{\infty} \frac{a_p}{z^p} \tag{A.17}$$

and call the right-hand side the asymptotic expression, expansion or representation of $f(z)$. The series itself may be either convergent of divergent (ABRAMOWITZ, 1972, p. 15).

Two different functions may have the same asymptotic expression, *i.e.* an asymptotic expansion is not unique (MORSE and FESHBACH, 1953, p. 436). For example the function e^{-z} can be added to the above-mentioned $f(z)/\varphi(z)$ without changing the asymptotic representation when $\mathrm{Re} z > 0$.

In this work we concern ourselves mainly with the first term of the asymptotic expansion, *i.e.* consider equation (A.16) in the form

$$\lim_{|z|\to\infty} \left(\frac{f(z)}{\varphi(z)} - 1 \right) = 0 \tag{A.18}$$

and write formula (A.17) as

$$f(z) \approx \varphi(z) \tag{A.19}$$

The coefficient a_0 is included in $\varphi(z)$.

A.6. Bessel, Neumann and Hankel functions

MAGNUS *et al.*, 1966, pp. 65–151, and OLVER, 1972, are general references for the subject of this section and the presentation below is mainly based on them.

The differential equation

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - \nu^2) f = 0 \qquad \text{(A.20)}$$

where z and ν are arbitrary complex numbers is called Bessel's differential equation. z is known as the argument and ν as the order. Special solutions of equation (A.20) are the Bessel function $J_\nu(z)$, the Neumann function $Y_\nu(z)$ and the Hankel functions of the first and of the second kind $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$. These functions are defined by

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}, \qquad \text{(A.21)}$$

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}, \qquad \text{(A.22)}$$

$$H_\nu^{(1)}(z) \doteq J_\nu(z) + iY_\nu(z) \qquad \text{(A.23)}$$

and

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z). \qquad \text{(A.24)}$$

The gamma function appearing in formula (A.21) can be expressed as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad \text{(A.25)}$$

for $\mathrm{Re}\, z > 0$ (see MAGNUS *et al.*, 1966, pp. 1–13). The Γ-function satisfies the equations

$$\Gamma(z+1) = z\Gamma(z), \qquad \text{(A.26)}$$

$$\Gamma(n+1) = n! \quad \text{(n non-negative integer)} \qquad \text{(A.27)}$$

and

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \qquad \text{(A.28)}$$

If ν is an integer, the right-hand side of equation (A.22) is replaced by its limiting value. Formulae (A.23) and (A.24) resemble the so-called Euler formulae

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta \qquad \text{(A.29)}$$

where θ may be real or complex.

The functions $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent for all values of ν. Hence every solution of equation (A.20) is expressible as a linear combination of these functions. It is seen from formulae (A.23) and (A.24) that $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ constitute another linearly independent pair.

For the mathematical theory of $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, WATSON, 1948, can be referred to. In this work we express that each is analytic, and so possesses all derivatives, throughout the complex z-plane cut along the negative real axis. The region $-\pi < \arg z < \pi$ is called the principal branch of the functions in question. Values for points z in other branches can be reduced to values in the principal branch using the following formulae, in which m is an arbitrary integer:

$$J_\nu(ze^{\mathrm{i}m\pi}) = e^{\mathrm{i}m\nu\pi} J_\nu(z), \tag{A.30}$$

$$Y_\nu(ze^{\mathrm{i}m\pi}) = e^{-\mathrm{i}m\nu\pi} Y_\nu(z) + 2\mathrm{i}\sin(m\nu\pi)\cot(\nu\pi)\, J_\nu(z), \tag{A.31}$$

$$H_\nu^{(1)}(ze^{\mathrm{i}m\pi}) = -\frac{\sin((m-1)\,\nu\pi)}{\sin(\nu\pi)}\, H_\nu^{(1)}(z) - e^{-\mathrm{i}\nu\pi}\,\frac{\sin(m\nu\pi)}{\sin(\nu\pi)}\, H_\nu^{(2)}(z) \tag{A.32}$$

and

$$H_\nu^{(2)}(ze^{\mathrm{i}m\pi}) = \frac{\sin((m+1)\nu\pi)}{\sin(\nu\pi)}\, H_\nu^{(2)}(z) + e^{i\nu\pi}\,\frac{\sin(m\nu\pi)}{\sin(\nu\pi)}\, H_\nu^{(1)}(z). \tag{A.33}$$

Hence especially

$$H_\nu^{(1)}(ze^{i\pi}) = -e^{-i\nu\pi} H_\nu^{(2)}(z) \tag{A.34}$$

and

$$H_\nu^{(2)}(ze^{-i\pi}) = -e^{i\nu\pi} H_\nu^{(1)}(z). \tag{A.35}$$

Equations

$$F_{\nu-1}(z) + F_{\nu+1}(z) = \frac{2\nu}{z}\, F_\nu(z), \tag{A.36}$$

$$F_{\nu-1}(z) - F_{\nu+1}(z) = 2\,\frac{dF_\nu(z)}{dz}, \tag{A.37}$$

$$\frac{dF_\nu(z)}{dz} = F_{\nu-1}(z) - \frac{\nu}{z} F_\nu(z) \tag{A.38}$$

and

$$\frac{dF_\nu(z)}{dz} = -F_{\nu+1}(z) + \frac{\nu}{z} F_\nu(z) \tag{A.39}$$

are known as recurrence relations. The symbol F denotes J, Y, $H^{(1)}$, $H^{(2)}$ or any linear combination of these the coefficients in which are independent of z and ν. For $\nu = 0$ equation (A.39) yields

$$\frac{dF_0(z)}{dz} = -F_1(z). \tag{A.40}$$

Other formulae:

$$J_{-n}(z) = (-1)^n J_n(z), \qquad \text{(n integer)} \tag{A.41}$$

$$Y_{-n}(z) = (-1)^n Y_n(z), \qquad \text{(n integer)} \tag{A.42}$$

$$H_{-\nu}^{(1)}(z) = e^{\nu\pi i} H_\nu^{(1)}(z), \tag{A.43}$$

$$H_{-\nu}^{(2)}(z) = e^{-\nu\pi i} H_\nu^{(2)}(z), \tag{A.44}$$

$$J_\nu(z)\, H_{\nu-1}^{(1)}(z) - J_{\nu-1}(z)\, H_\nu^{(1)}(z) = \frac{2}{i\pi z}, \tag{A.45}$$

$$J_\nu(z)\, H_{\nu-1}^{(2)}(z) - J_{\nu-1}(z)\, H_\nu^{(2)}(z) = -\frac{2}{i\pi z}. \tag{A.46}$$

When ν is fixed and z approaches zero, the following approximations are valid:

$$J_\nu(z) \approx \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \qquad (\nu \neq -1, -2, \ldots), \tag{A.47}$$

$$Y_0(z) \approx -iH_0^{(1)}(z) \approx iH_0^{(2)}(z) = \frac{2}{\pi} \log z \tag{A.48}$$

and

$$Y_\nu(z) \approx -iH_\nu^{(1)}(z) \approx iH_\nu^{(2)}(z) = -\frac{\Gamma(\nu)}{\pi} \left(\frac{2}{z}\right)^\nu \; (Re\,\nu > 0). \tag{A.49}$$

When ν is fixed and $|z|$ approaches infinity, the following asymptotic expressions can be used

$$J_\nu(z) = \left(\frac{\pi z}{2}\right)^{-1/2} (P(\nu,z)\cos X - Q(\nu,z)\sin X), \quad -\pi < \arg z < \pi, \tag{A.50}$$

$$Y_\nu(z) = \left(\frac{\pi z}{2}\right)^{-1/2} (P(\nu,z)\sin X + Q(\nu;z)\cos X), \quad -\pi < \arg z < \pi, \tag{A.51}$$

$$H_\nu^{(1)}(z) = \left(\frac{\pi z}{2}\right)^{-1/2} (P(\nu,z) + iQ(\nu,z))\, e^{iX}, \qquad -\pi < \arg z < 2\pi, \tag{A.52}$$

and

$$H_\nu^{(2)}(z) = \left(\frac{\pi z}{2}\right)^{-1/2} (P(\nu,z) - iQ(\nu,z))\, e^{-iX}, \qquad -2\pi < \arg z < \pi, \tag{A.53}$$

where

$$X = z - \frac{\nu\pi}{2} - \frac{\pi}{4}, \tag{A.54}$$

$$P(\nu,z) \simeq 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(\nu,2k)}{(2z)^{2k}} \tag{A.55}$$

and

$$Q(\nu,z) \simeq \sum_{k=0}^{\infty} (-1)^k \frac{(\nu,2k+1)}{(2z)^{2k+1}}. \tag{A.56}$$

The symbol (ν,m) is defined by

$$(\nu,m) = \frac{1}{2^{2m}m!} \prod_{k=1}^{m} (4\nu^2 - (2k-1)^2). \tag{A.57}$$

By taking only the first terms in the series, the Hankel functions for large values of $|z|$ can be approximated as follows

$$H_\nu^{(1)}(z) \approx \left(\frac{\pi z}{2}\right)^{-1/2} e^{i(z - \nu\pi/2 - \pi/4)}, \qquad -\pi < \arg z < 2\pi, \tag{A.58}$$

and

$$H_\nu^{(2)}(z) \approx \left(\frac{\pi z}{2}\right)^{-1/2} e^{-i(z - \nu\pi/2 - \pi/4)}, \qquad -2\pi < \arg z < \pi, \tag{A.59}$$

(*cf.* formulae (A.18) and (A.19)).

The differential equation

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} - (z^2 + \nu^2) f = 0, \tag{A.60}$$

where z is the complex argument and ν the complex order, is called the modified Bessel differential equation. Its linearly independent solutions $I_\nu(z)$ and $K_\nu(z)$ are related to the Bessel, Neumann and Hankel functions by the following formulae:

$$I_\nu(z) = e^{-i\nu\pi/2} J_\nu(ze^{i\pi/2}), \qquad -\pi < \arg z \leqslant \frac{\pi}{2}, \tag{A.61}$$

$$I_\nu(z) = e^{3i\nu\pi/2} J_\nu(ze^{-3i\pi/2}), \qquad \frac{\pi}{2} < \arg z \leqslant \pi, \tag{A.62}$$

$$K_\nu(z) = \frac{i\pi}{2} e^{i\nu\pi/2} H_\nu^{(1)}(ze^{i\pi/2}), \qquad -\pi < \arg z \leqslant \frac{\pi}{2}, \tag{A.63}$$

$$K_\nu(z) = -\frac{i\pi}{2} e^{-i\nu\pi/2} H_\nu^{(2)}(ze^{-i\pi/2}), \qquad -\frac{\pi}{2} < \arg z \leqslant \pi, \tag{A.64}$$

and

$$Y_\nu(ze^{i\pi/2}) = e^{i(\nu+1)\pi/2} I_\nu(z) - \frac{2}{\pi} e^{-i\nu\pi/2} K_\nu(z), \qquad -\pi < \arg z \leqslant \frac{\pi}{2}. \tag{A.65}$$

If x is real and positive and $-1/2 < Re\nu < 1/2$, $J_\nu(x)$ and $Y_\nu(x)$ can be expressed as follows:

$$J_\nu(x) = \frac{2}{\Gamma(1/2-\nu)\sqrt{\pi}} \left(\frac{x}{2}\right)^{-\nu} \int_1^\infty \frac{\sin(xt)}{(t^2-1)^{\nu+1/2}}\, dt \tag{A.66}$$

and

$$Y_\nu(x) = -\frac{2}{\Gamma(1/2-\nu)\sqrt{\pi}} \left(\frac{x}{2}\right)^{-\nu} \int_1^\infty \frac{\cos(xt)}{(t^2-1)^{\nu+1/2}}\, dt. \tag{A.67}$$

Let us now mention other formulae used in this work:

$$\int_0^\infty \frac{e^{-\alpha\sqrt{1+u^2}}}{\sqrt{1+u^2}} \cos(\gamma u)\cosh(\nu \operatorname{arsinh} u)\, du \tag{A.68}$$

$$= \int_0^\infty e^{-\alpha\cosh g} \cos(\gamma \sinh g)\cosh(\nu g)\, dg = \cos\left(\nu \overline{\arctan}\, \frac{\gamma}{\alpha}\right) K_\nu((\alpha^2+\gamma^2)^{1/2}).$$

According to MAGNUS *et al.*, 1966, p. 86, the latter equality requires that $Re(\alpha \pm i\gamma) > 0$. However, using formula 868 on page 111 of CAMPBELL and FOSTER, 1954, it can be shown that for $\nu = 0$ the case $Re\alpha = Im\gamma = 0$ is also acceptable. But in order not to make the argument of K_ν zero, we must assume that $\alpha \neq \pm i\gamma$ (see formulae (A.48), (A.49), (A.63) and (A.64)). Owing to the fact that the integral $\int_0^\infty \frac{\sin x}{x}\, dx = \pi/2$ exists it is obviously possible in the case $\nu = 0$ to extend the conditions for the validity of formula (A.68), even if $Re\alpha \neq 0$ and $Im\gamma \neq 0$, and express the requirements as $Re(\alpha \pm i\gamma) \geqslant 0$ and $\alpha \neq \pm i\gamma$.

We pointed out in Section A.2 that it is always important to specify the complex half plane where a square root is situated. The square roots in the first integrand of formula (A.68) are real and positive. According to MAGNUS *et al.*, 1966, p. 493, the quantity $(\alpha^2+\gamma^2)^{1/2}$ satisfies the conditions $-\pi/2 < \arg(\alpha^2+\gamma^2)^{1/2} < \pi/2$, unless $\alpha^2+\gamma^2$ is real and negative. Owing to the condition $Re(\alpha \pm i\gamma) > 0$, $\alpha^2+\gamma^2$ cannot, however, be a non-positive real number. This is also true, if $Re(\alpha \pm i\gamma) \geqslant 0$, $Im\gamma \neq 0$ and $\alpha \neq \pm i\gamma$. The conventions of CAMPBELL and FOSTER, 1954, on their pages 30 and 33, imply that $-\pi/2 < \arg(\alpha^2+\gamma^2)^{1/2} \leqslant \pi/2$ in the case $\nu = Im\gamma = 0$.

We obtain from formula 563.4 on page 61 of CAMPBELL and FOSTER, 1954:

$$\int_{-\infty}^{\infty} \frac{e^{i2\pi fg}\, df}{(i2\pi f)^{1/2}(i2\pi f+\rho)^{1/2}} = \begin{cases} e^{-\rho g/2} I_0(\rho g/2), & g > 0 \\ 0, & g < 0. \end{cases} \tag{A.69}$$

The arguments of both square roots are in the range $-\pi/2 < \arg \leqslant \pi/2$, and ρ is any non-zero complex number with a non-negative real part. The denominator of the integrand can be expressed as $i2\pi(f^2 - i\frac{\rho}{2\pi}f)^{1/2}$, but in order to keep the argument of the denominator correct, the quantity $(f^2 - i\frac{\rho}{2\pi}f)^{1/2}$ has to satisfy the conditions $-\pi \leqslant \arg(f^2 - i\frac{\rho}{2\pi}f)^{1/2} \leqslant -\pi/2$ for $f < 0$ and $-\pi/2 \leqslant \arg(f^2 - i\frac{\rho}{2\pi}f)^{1/2} \leqslant 0$ for $f > 0$. So

$$\int_{-\infty}^{\infty} \frac{ie^{i2\pi fg}df}{\left(f^2 - i\frac{\rho}{2\pi}f\right)^{1/2}} = -2\pi e^{-\rho g/2} J_0\left(i\frac{\rho g}{2}\right)\theta(g), \tag{A.70}$$

where formulae (A.5) and (A.61) have also been utilized. The case $g = 0$, which yields $-\pi$, *i.e.* half of the limits $g \to 0+$ and $g \to 0-$, can be calculated separately.

The employment of formula 917 on page 125 of Campbell and Foster, 1954, shows that

$$\int_{-\infty}^{\infty} K_0(\sigma(\rho^2 + 4\pi^2 f^2)^{1/2})e^{i2\pi fg}df = \frac{e^{-\rho(g^2+\sigma^2)^{1/2}}}{2(g^2 + \sigma^2)^{1/2}} \tag{A.71}$$

where $\sigma(\neq 0)$ and ρ are any complex numbers with non-negative real parts and g is real. The arguments of the square roots are in the range $-\pi/2 < \arg \leqslant \pi/2$. Let us change the variable of integration from f to $u = -2\pi f$ and assume that σ is expressible as $\beta e^{i\pi/2} = i\beta$ where $\beta \neq 0$ and $-\pi/2 \leqslant \arg\beta \leqslant 0$. Formula (A.64) can now be used, and we obtain:

$$\int_{-\infty}^{\infty} H_0^{(2)}(\beta(\rho^2 + u^2)^{1/2})e^{-iug}du = 2i\,\frac{e^{-\rho(g^2-\beta^2)^{1/2}}}{(g^2 - \beta^2)^{1/2}} \tag{A.72}$$

Since $H_0^{(2)}(\beta(\rho^2 + u^2)^{1/2}$ is even with respect to u, the factor e^{-iug} can be replaced by e^{iug} or by $\cos ug$. In the latter case the contributions from $-\infty$ to 0 and from 0 to ∞ are equal.

The inverse formula of equation (A.71) is also obtained from Campbell's and Forster's formula 917:

$$\int_{-\infty}^{\infty} \frac{e^{-\rho(g^2+\sigma^2)^{1/2}}}{2(g^2 + \sigma^2)^{1/2}}\, e^{-i2\pi fg}dg = K_0(\sigma(\rho^2 + 4\pi^2 f^2)^{1/2}) \tag{A.73}$$

where the arguments of the square roots are in the range $-\pi/2 < \arg \leqslant \pi/2$, $\sigma(\neq 0)$

and ρ are arbitrary complex numbers with non-negative real parts. The validity of equations (A.71) and (A.73) is also seen from the above-mentioned formula 868 of Campbell and Foster. By writing $-2\pi f$ as u and by assuming that $\sigma = i\beta$ where $\beta \neq 0$ and $-\pi/2 \leqslant \arg\beta \leqslant 0$ equation (A.73) yields the inverse of formula (A.72):

$$\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{e^{-\rho(g^2-\beta^2)^{1/2}}}{(g^2-\beta^2)^{1/2}} e^{iug} dg = H_0^{(2)}(\beta(\rho^2+u^2)^{1/2}) . \tag{A.74}$$

Analogously to equation (A.72), the factor e^{iug} can be replaced by e^{-iug} or by $\cos ug$ in formula (A.74), and in the latter case the contributions from $-\infty$ to 0 and from 0 to ∞ are equal.

By deriving equations (A.72) and (A.74) with respect to ρ, and by utilizing formula (A.40) for $H^{(2)}$ we obtain

$$\beta\rho \int_{-\infty}^{\infty} \frac{H_1^{(2)}(\beta(\rho^2+u^2)^{1/2})}{(\rho^2+u^2)^{1/2}} e^{-iug} du = 2ie^{-\rho(g^2-\beta^2)^{1/2}} \tag{A.75}$$

and

$$\frac{i}{\pi} \int_{-\infty}^{\infty} e^{-\rho(g^2-\beta^2)^{1/2}} e^{iug} dg = \beta\rho \frac{H_1^{(2)}(\beta(\rho^2+u^2)^{1/2})}{(\rho^2+u^2)^{1/2}} . \tag{A.76}$$

These formulae are also obtained from formula 917.5 on page 125 of Campbell and Foster, 1954. In addition to the conditions of equations (A.72) and (A.74) their validity requires that the real part of ρ be positive. But on the other hand, the case $\beta = 0$ can be included in equations (A.75) and (A.76) (see formula (A.49)).

The equation

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - \nu^2) f = \frac{4}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} \tag{A.77}$$

is known as the inhomogeneous Bessel differential equation. The so-called Struve function

$$H_\nu(z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{z}{2}\right)^{\nu+2m+1} \left[\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(\nu+m+\frac{3}{2}\right)\right]^{-1} \tag{A.78}$$

satisfies equation (A.77). The Struve function has the asymptotic representation

$$H_\nu(z) \simeq Y_\nu(z) + \frac{1}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+\nu\right)}\left(\frac{z}{2}\right)^{\nu-1} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}-\nu\right)_m (2m)!}{m!\, z^{2m}} \tag{A.79}$$

for $-\pi < \arg z < \pi$. In this formula $Y_\nu(z)$ is the Neumann function and

$$\left(\frac{1}{2}-\nu\right)_m = \left(\frac{1}{2}-\nu\right)\left(\frac{1}{2}-\nu+1\right)\left(\frac{1}{2}-\nu+2\right)\cdots\left(\frac{1}{2}-\nu+m-1\right) \tag{A.80}$$

For m = 0 this quantity is equal to one. According to the last formula on page 440 of MAGNUS et al., 1966,

$$H_\nu(as) - Y_\nu(as) = \frac{1}{2^{\nu-1}\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+\nu\right)}\left(\frac{s}{a}\right)^{\nu} \int_0^{\infty} e^{-st}(a^2+t^2)^{\nu-1/2}\,dt \tag{A.81}$$

where a is any complex number and the real part of s is positive. In order that $a^2 + t^2$ is not real and negative let us assume that the real part of a is non-zero (see MAGNUS *et al.,* 1966, p. 493).

Appendix B. Basic electromagnetic theory

The basic electromagnetic theory has been presented by STRATTON, 1941, HARNWELL, 1949, SOMMERFELD, 1959, FEYNMAN *et al.*, 1964, JONES, 1964, PANOFSKY and PHILLIPS, 1964, JACKSON, 1975, and LIPAS, 1972. It is, however, included also in this work because the classical electromagnetic principles are the basis of the treatments in Chapters 2, 3 and 4.

B.1. Maxwell's equations

The classical electromagnetic theory is based on Maxwell's equations

$$\nabla\cdot\bar{E} = \frac{\rho}{\epsilon_0}, \tag{B.1}$$

$$\nabla\cdot\bar{B} = 0, \tag{B.2}$$

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} \tag{B.3}$$

and

$$\nabla\times\bar{B} = \mu_0\bar{j} + \mu_0\epsilon_0\frac{\partial\bar{E}}{\partial t}. \tag{B.4}$$

They couple the electric field (intensity) $\bar{E}(\bar{r},t)$ (V/m), the magnetic flux density $\bar{B}(\bar{r},t)$ (or the magnetic field) (Vs/m^2), the electric charge density $\rho(\bar{r},t)$ (C/m^3) and the electric current density $\bar{j}(\bar{r},t)$ (A/m^2) together. $\bar{r}$ is the position vector and t is the time. The quantities ϵ_0 and μ_0 are constants, and they are called the permittivity and the permeability of free space, respectively: $\epsilon_0 \approx 8.854\cdot 10^{-12}$ As/Vm, $\mu_0 = 4\pi\cdot 10^{-7}$ Vs/Am. The speed of light in free space is $c = 1/\sqrt{\mu_0\epsilon_0} = 2.998\cdot 10^8$ m/s.

It is customary to divide ρ and $\bar{j}$, which are called sources, into different types of charges and currents by introducing polarization and magnetization vectors $\bar{P}$ and $\bar{M}$. The polarization charge ρ_P, the polarization current $\bar{j}_P$ and the magnetization current $\bar{j}_M$ are defined as follows:

$$\rho_P = -\nabla\cdot\bar{P}, \tag{B.5}$$

$$\bar{j}_P = \frac{\partial\bar{P}}{\partial t} \quad \text{and} \tag{B.6}$$

$$\bar{j}_M = \nabla\times\bar{M}. \tag{B.7}$$

The vectors $\bar{P}$ and $\bar{M}$ could in principle be any vector fields, but as usual they are regarded as the volume densities of electric and magnetic dipoles, respectively.

If further two vectors, the electric displacement vector or the electric flux density $\bar{D}$ and the magnetic field intensity $\bar{H}$, are defined by

$$\bar{D} = \epsilon_0 \bar{E} + \bar{P} \tag{B.8}$$

and

$$\bar{H} = \frac{1}{\mu_0} \bar{B} - \bar{M}, \tag{B.9}$$

Maxwell's equations can be written as

$$\nabla \cdot \bar{D} = \rho_{true}, \tag{B.10}$$

$$\nabla \cdot \bar{B} = 0, \tag{B.11}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{B.12}$$

and

$$\nabla \times \bar{H} = \bar{j}_{true} + \frac{\partial \bar{D}}{\partial t} \tag{B.13}$$

where

$$\rho_{true} = \rho - \rho_P \tag{B.14}$$

and

$$\bar{j}_{true} = \bar{j} - \bar{j}_P - \bar{j}_M. \tag{B.15}$$

So ρ_{true} and $\bar{j}_{true}$ simply represent the charges and the currents which are left over after ρ_P and $\bar{j}_P + \bar{j}_M$, as defined by formulae (B.5) – (B.7), are subtracted from the total charge and from the total current. The following interpretation can be made: the true charges and currents refer to the charges which move freely in the medium, while ρ_P, $\bar{j}_P$ and $\bar{j}_M$ are associated with charges which are bound and can only be detected in an atomic scale. This division of the total charge is a macroscopic procedure. Microscopically there is only one kind of charge. Hence equations (B.1) – (B.4) are always the correct Maxwell equations, and the equivalent equations (B.10) – (B.13), with equations (B.5) – (B.9), (B.14) and (B.15), are suitable for macroscopic treatments.

Let us consider Maxwell's equations (B.1) – (B.4) valid at every point and time where the space variations and time variations of the sources $\rho(\bar{r}, t)$ and $\bar{j}(\bar{r}, t)$ are continuous, and presume further that the sources are finite at these points and at these times. The fields $\bar{E}(\bar{r}, t)$ and $\bar{B}(\bar{r}, t)$ and their derivatives appearing in Maxwell's equations are also assumed to have the same properties, *i.e.* the continuity and finiteness at the points and times in question. The finiteness of $\bar{E}$ and $\bar{B}$ is natural for physical reasons. The time continuity of physical phenomena also seems clear. Space discontinuities in electromagnetism are discussed in Section B.7. Similar statements are obtained for Maxwell's equations (B.10) – (B.13) if ρ and $\bar{j}$ are replaced by ρ_{true} and $\bar{j}_{true}$ and the assumptions about $\bar{E}$ and $\bar{B}$ are extended to $\bar{D}$ and $\bar{H}$.

Normally in the discussion of electromagnetic problems, as in this work »sufficient» mathematical regularities of the quantities are implicitly or explicitly assumed. For example at the end of this section and in Section B.3 certain divergences are assumed to exist and the order of derivation is changed.

It is common to use Dirac delta functions (Section A.3) in mathematical expressions of idealized electromagnetic sources, *e.g.* a point charge. From the viewpoint of the above-assumed continuity and finiteness of the sources Maxwell's equations can not in principle be used at the non-zero points of the delta functions. The difficulty is avoided if the source is actually considered continuous and finite but to be so close to the delta function in question that the mathematical properties of delta functions may be used. Thus we simply believe that the use of delta functions in electromagnetic calculations is correct. The »strength» of the infinity associated with a delta function is known exactly and, when multiplied by a differential, gives unity (see formulae (A.3) and (A.4)).

The divergence of equation (B.3) establishes that the divergence of the magnetic field is constant in time. If it is assumed that this divergence has vanished at some time in the past (or will vanish in the future), the constant has to be zero, and equation (B.2) has been derived from equation (B.3). The vanishing of the divergence is achieved for example by assuming that before a certain moment the magnetic field was zero everywhere in space.

B.2. Lorentz force

The force acting on a point charge q situated in an electromagnetic field $\bar{E}$, $\bar{B}$ is

$$\bar{F} = q(\bar{E} + \bar{v} \times \bar{B}) \tag{B.16}$$

where $\bar{v}$ is the velocity of the charge. It is seen that the work per unit time done by the field to the charge is

$$\frac{dW}{dt} = \bar{F} \cdot \bar{v} = q\bar{E} \cdot \bar{v}. \tag{B.17}$$

The magnetic field does no work because the magnetic force in formula (B.16) is perpendicular to $\bar{v}$. Equation (B.16), which is called the Lorentz force, and Maxwell's equations include the classical electromagnetic theory.

B.3. Equation of continuity

The divergence of Maxwell's equation (B.4) yields $0 = \nabla \cdot \bar{j} + \epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \bar{E}$. By substituting ρ/ϵ_0 from (B.1) for $\nabla \cdot \bar{E}$ this can be written as

$$\nabla \cdot \bar{j} + \frac{\partial \rho}{\partial t} = 0. \tag{B.18}$$

This equation, which expresses the conservation of charge, is known as the equation of continuity. Equation (B.18) was derived for the total charge and the total current, but it is equally valid for the true charge and the true current only. It is also valid for the polarization charge and the polarization current, as well as for the magnetization current. No magnetization charge exists.

»Inversely» to the above derivation of formula (B.18) it is seen that Maxwell's equation (B.4) and the equation of continuity imply the quantity $\nabla \cdot \bar{E} - \rho/\epsilon_0$ to be constant in time. Hence, if the vanishing of $\nabla \cdot \bar{E} - \rho/\epsilon_0$ at some moment can be assumed, equation (B.1) is obtained. This result is exactly analogous to the above-mentioned relationship between Maxwell's equations (B.2) and (B.3).

The quantity $\epsilon_0(\partial\bar{E}/\partial t)$, which does not represent actual current, has the dimension of electric current density and is called the vacuum displacement current. Equation (B.4) shows that the sum of the total current and the vacuum displacement current is non-divergent. The vector $\partial\bar{D}/\partial t$, which is equal to the vacuum displacement current plus the polarization current, is called the (total) displacement current.

B.4. Constitutive equations

The solution of equations (B.10) – (B.13) needs other relationships between the vectors $\bar{D}$, $\bar{B}$, $\bar{E}$, $\bar{H}$ and $\bar{j}_{true}$. Such relationships, which are called constitutive equations, depend on the structure of the matter in question and thus belong to the field of solid state physics. In the simplest case, however, the relationships between $\bar{P}$ and $\bar{E}$ and between $\bar{M}$ and $\bar{H}$ can be considered linear:

$$\bar{P} = \chi_e \epsilon_0 \bar{E} \tag{B.19}$$

and

$$\bar{M} = \chi_m \bar{H} \tag{B.20}$$

with scalar proportionality factors χ_e and χ_m, which are called the electric and the magnetic susceptibility, respectively, constant in time and space. From equations (B.8), (B.9), (B.19) and (B.20) we obtain:

$$\bar{D} = \epsilon \bar{E} \tag{B.21}$$

and

$$\bar{H} = \frac{1}{\mu} \bar{B}, \tag{B.22}$$

where $\epsilon = (1 + \chi_e)\,\epsilon_0$ is the permittivity or the dielectric constant of the medium and $\mu = (1 + \chi_m)\,\mu_0$ is the permeability of the medium. ϵ and μ are always (considered) positive. In free space $\chi_e = \chi_m = 0$ so that $\epsilon = \epsilon_0$ and $\mu = \mu_0$.

If equations (B.21) and (B.22) are valid, equations (B.10) and (B.13) become

$$\nabla \cdot \bar{E} = \frac{\rho_{true}}{\epsilon} \tag{B.23}$$

and

$$\nabla \times \bar{B} = \mu \bar{j}_{true} + \mu\epsilon \frac{\partial \bar{E}}{\partial t}. \tag{B.24}$$

In many cases the current density $\bar{j}_{true}$ and the electric field $\bar{E}$ are proportional, with a non-negative scalar proportionality factor σ constant in time and space, so that

$$\bar{j}_{true} = \sigma \bar{E}. \tag{B.25}$$

This equation is known as Ohm's law, and σ is the conductivity of the medium. In free space $\sigma = 0$. Let us call a medium which satisfies formulae (B.19) – (B.22) and (B.25) with time- and space-constants ϵ, μ and σ »simple». The constancy of ϵ and μ is equivalent to the constancy of χ_e and χ_m, respectively.

In accordance with the equation of continuity (B.18) (for the true charge and current) and with formulae (B.23) and (B.25)

$$\frac{\partial \rho_{true}}{\partial t} + \frac{\sigma}{\epsilon} \rho_{true} = 0 \tag{B.26}$$

or

$$\rho_{true} = \rho_0 \, e^{-(\sigma/\epsilon)\,t} \tag{B.27}$$

the constant of integration ρ_0 being equal to the true charge density at the point in question at the time $t=0$. The time $\tau = \epsilon/\sigma$ is called the relaxation time of the material. For conductors τ is extremely small, for example for copper $\tau \approx 2 \cdot 10^{-19}$s. Equation (B.27) was derived to the true charge, but according to formulae (B.1) and (B.23) and to formulae (B.5), (B.19) and (B.23) the total charge density as well as the polarization charge density are proportional to ρ_{true} in a »simple» medium (see below). So all kinds of initial charge density vanish exponentially with time (if $\sigma > 0$), and remain zero at every point of the medium where they are initially zero. Since charge is conserved, the disappearing charge has to appear somewhere else. Now the nearest possible points where the appearance can take place are situated at the surface of the medium. The surface charge must appear at the moment the exponential decay of the charge density begins inside the medium. Such simultaneity, however, seems to contradict the fact that no signal can exceed the speed of light. The reason for this contradiction probably lies in the assumption that Ohm's law (B.25), which is approximate, is valid.

From equations (B.1), (B.5) – (B.7), (B.19), (B.20) and (B.22) – (B.25) it can be shown that the charge and current densities satisfy the following equations in a »simple» medium

$$\rho_P = -\chi_e \epsilon_0 \nabla \cdot \bar{E} = -\chi_e \rho = \left(1 - \frac{\epsilon}{\epsilon_0}\right) \rho , \tag{B.28}$$

$$\rho_{true} = \epsilon \nabla \cdot \bar{E} = \frac{\epsilon}{\epsilon_0} \rho = \frac{\epsilon}{\epsilon_0 - \epsilon} \rho_P , \tag{B.29}$$

$$\rho = \frac{\epsilon_0}{\epsilon} \rho_{true} \, (= \rho_{true} + \rho_P), \tag{B.30}$$

$$\bar{j}_{true} = \sigma \bar{E} , \tag{B.31}$$

$$\bar{j}_P = \chi_e \epsilon_0 \frac{\partial \bar{E}}{\partial t} = (\epsilon - \epsilon_0) \frac{\partial \bar{E}}{\partial t} = \frac{\epsilon - \epsilon_0}{\sigma} \frac{\partial \bar{j}_{true}}{\partial t} , \tag{B.32}$$

$$\bar{j}_M = \chi_m \nabla \times \bar{H} = \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) \nabla \times \bar{B} = \left(\frac{\mu}{\mu_0} - 1\right) \left(\bar{j}_{true} + \epsilon \frac{\partial \bar{E}}{\partial t}\right) \tag{B.33}$$

$$= \left(\frac{\mu}{\mu_0} - 1\right) \left(\sigma \bar{E} + \epsilon \frac{\partial \bar{E}}{\partial t}\right) = \left(\frac{\mu}{\mu_0} - 1\right) \left(\bar{j}_{true} + \frac{\epsilon}{\sigma} \frac{\partial \bar{j}_{true}}{\partial t}\right)$$

and

$$\bar{j} = \bar{j}_{true} + \bar{j}_P + \bar{j}_M = \frac{\mu}{\mu_0} \bar{j}_{true} + \left(\frac{\mu \epsilon}{\mu_0 \sigma} - \frac{\epsilon_0}{\sigma}\right) \frac{\partial \bar{j}_{true}}{\partial t} . \tag{B.34}$$

The definitions of the permittivity and the permeability have also been utilized. The true charge density has already been calculated and is given by formula (B.27).

The situation is, however, more complicated if there are »external» charges and currents in addition to the »internal» charges and currents associated with the field itself and expressed in formulae (B.27)–(B.34). The effect of such »external» charges and currents can evidently be taken into account in the treatment by changing the true charge and the true current in Maxwell's equations (B.23) and (B.24) suitably. Then formulae (B.28) – (B.30) (but not (B.27)) are satisfied by the sums of the »internal» and »external» charges, and no such division of the charge is actually to be made. Also in the case of equations (B.27) – (B.30) the word »internal» may be misleading; for example, the initial charge ρ_0 in formula (B.27) could just as well be called »external».

Let us point out that if it is assumed that all existing currents are given by formulae (B.31) – (B.34), formula (B.24) (with $\bar{j}_{true} = \sigma\bar{E}$) is directly obtained from Maxwell's equation (B.4). Equation (B.24) combined with Maxwell's equation (B.1) then implies that the total charge density satisfies formula (B.27).

If the above-mentioned definition of a »simple» medium were taken exactly, the concept of »external» quantities introduced would not be allowable. Let us, however, accept this concept, and for example an »external» true current $\bar{j}'_{true}$ is made to obey Ohm's law by introducing an equivalent »external» electric field $\bar{E}'$ such that $\bar{j}'_{true} = \sigma\bar{E}'$.

More generally, the permittivity, the permeability and the conductivity need not be scalars but they may also be tensors, which implies anisotropy. However, in this work they are everywhere considered scalars. Further, these parameters in question need not be constants in space and time, although the latter is assumed throughout this work. The parameters also depend on the frequency with which the fields vary in time. This dependence is neglected here, too, which is permissible for low frequencies (below 10^8 Hz, STRATTON, 1941, pp. 321 and 327). On the other hand when considering only one frequency, it is formally insignificant whether the parameters are frequency-dependent or not. The use of permittivity, permeability and conductivity scalars or tensors implies linearity. Nonlinear cases, which are not treated in this work, are also possible.

B.5. Harmonic time-dependence of electromagnetic fields in »simple» media

Let us now assume that the time-dependence of the electromagnetic field is harmonic, *i.e.* $e^{i\omega t}$ (see Section A.1), and consider a »simple» medium characterized by the parameters ϵ, μ and σ. Assume also that there are no currents of »external» origin, which means that the only currents appearing are those given by equations (B.31) – (B.34). Maxwell's equations (B.1) – (B.4) can then be written as

$$\nabla \cdot \bar{E} = 0, \tag{B.35}$$

$$\nabla \cdot \bar{B} = 0, \tag{B.36}$$

$$\nabla \times \bar{E} = -i\omega\bar{B} \tag{B.37}$$

and

$$\nabla \times \bar{B} = \mu(\sigma + i\omega\epsilon)\bar{E}. \tag{B.38}$$

Equation (B.38) can be obtained directly from formula (B.24). The quantity ω could, of course, be any arbitrary constant in time in equations (B.35)–(B.38), but the word »harmonic« implies that it is real and time- and space-constant.

No assumption was made about the charge density. However, equation (B.35) states that ρ is zero. This is due to equation (B.38), which makes formula (B.35) valid if either σ or ω differs from zero. In fact, equation (B.35) follows from the equation $\nabla \times \bar{H} = (\sigma + i\omega\epsilon)\bar{E}$ obtained from formula (B.13), so that no assumption about μ is actually necessary. The fact that the only possible time-dependence of the total charge density, under the assumptions made, is given by formula (B.27) can also be referred to here. It can also be seen that equation (B.36) results from equation (B.37) (assuming that ω is not zero). This is a direct consequence of the more general statement made in Section B.1 about the relationship between formulae (B.2) and (B.3).

The curls of equations (B.37) and (B.38), with the use of the other equations (B.35) – (B.38), give the so-called wave equations for harmonically time-dependent electromagnetic fields

$$\nabla^2\bar{E} - i\omega\mu\sigma\bar{E} + \omega^2\mu\epsilon\bar{E} = 0 \tag{B.39}$$

and

$$\nabla^2\bar{B} - i\omega\mu\sigma\bar{B} + \omega^2\mu\epsilon\bar{B} = 0. \tag{B.40}$$

Unless the harmonic time-dependence is assumed $i\omega$ must be replaced by $\partial/\partial t$ and ω^2 by $-\partial^2/\partial t^2$ in equations (B.39) and (B.40) to obtain the general wave equations. The vanishing of the charge density must be assumed to obtain formula (B.35), if the time dependence is arbitrary. In non-conducting media the second terms of equations (B.39) and (B.40) vanish, and the equations become Helmholtz equations. In a well-conducting medium the third terms are usually insignificant compared with the second terms and equations (B.39) and (B.40) reduce to the so-called diffusion equations with a harmonic time-dependence.

Let us define the complex propagation constant k with

$$k^2 = \omega^2\mu\epsilon - i\omega\mu\sigma. \tag{B.41}$$

This equation does not define k unambiguously unless k is restricted to a half-plane, for instance $-\pi/2 < \arg k \leqslant \pi/2$ (see Section A.2). Assume now that the quantity ω is non-negative, being the normal angular frequency. Then, considering the restriction of $\arg k$ and equation (B.41), we see that

$$-\frac{\pi}{4} \leqslant \arg k \leqslant 0. \tag{B.42}$$

Using the propagation constant, equations (B.39) and (B.40) can be written simply as

$$\nabla^2\bar{E} + k^2\bar{E} = 0 \tag{B.43}$$

and

$$\nabla^2\bar{B} + k^2\bar{B} = 0. \tag{B.44}$$

The wave equations (B.43) and (B.44) result from Maxwell's equations (B.35) – (B.38), but an arbitrary solution of equations (B.43) and (B.44) need not satisfy equations (B.35) – (B.38).

It was mentioned above that in a well-conducting medium the second terms are normally much larger than the third terms in equations (B.39) and (B.40). Hence, in a conducting medium, k can be accurately approximated by $(1-i)/\delta$, where the non-negative quantity

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{B.45}$$

is known as the skin depth and has the dimension of length. The skin depth measures the depth of penetration of an electromagnetic field into a conductor. The skin depth approaches zero, if the conductivity goes to infinity. Moreover δ is also smaller the higher the frequency, but the approximation of k with $(1-i)/\delta$ requires that $\epsilon\omega \ll \sigma$. If this condition is not valid, k must be written exactly as

$$k = \omega\sqrt{\mu\epsilon}\left(\sqrt{\frac{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1}{2}} - i\sqrt{\frac{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1}{2}}\right) \tag{B.46}$$

where all square roots are non-negative.

B.6. Electromagnetic plane waves

In the case of electromagnetic plane waves the fields are defined to depend only on one space coordinate and on time. Let us further assume, as is usual in the discussion of plane waves, that the field appears in a »simple» medium without any kind of charges or »external» currents and also that the time-dependence of the electromagnetic field is harmonic $e^{i\omega t}$ with a positive angular frequency ω. The discussion of Section B.5 is thus directly applicable, and the vanishing of the total charge is actually a consequence of the assumptions concerning the medium, the currents and the time-dependence.

If the single space-coordinate upon which the fields of a plane wave depend is z, equation (B.43) takes the form

$$\frac{d^2\bar{E}(z)}{dz^2} + k^2\bar{E}(z) = 0. \tag{B.47}$$

The time factor $e^{i\omega t}$ has been divided out of this equation. Equation (B.47) can easily be solved and the following expression is obtained for the electric field

$$\bar{E}(z,t) = \bar{E}(z)e^{i\omega t} = \bar{E}_0 e^{i(\omega t-kz)} + \bar{E}_1 e^{i(\omega t+kz)} \tag{B.48}$$

where $\bar{E}_0$ and $\bar{E}_1$ are complex constant vectors. The magnetic field can be calculated from equation (B.37) and

$$\bar{B}(z,t) = \frac{k}{\omega}\,\hat{e}_z \times \bar{E}_0 e^{i(\omega t-kz)} - \frac{k}{\omega}\,\hat{e}_z \times \bar{E}_1 e^{i(\omega t+kz)} \tag{B.49}$$

where $\hat{e}_z$ is the unit vector in the direction of the positive z-axis. Equation (B.37) is thus satisfied making also equation (B.36) valid. In order for equation (B.38) to be true, the z-component of $\bar{E}$ must be zero, and so the z-components of both $\bar{E}_0$ and $\bar{E}_1$ are set equal to zero. Thus equation (B.35) is satisfied. The complete validity of equation (B.38) now results from the satisfaction of formulae (B.35), (B.37) and (B.43), and so the plane wave electromagnetic field expressed by equations (B.48) and (B.49) satisfies all Maxwell's equations (B.35) – (B.38).

Since according to formula (B.49) the z-component of $\bar{B}$ is also zero, harmonic plane waves are transverse with respect to both the electric and the magnetic fields. If no assumption of the time-dependence of the field is made, plane waves may in principle have non-zero longitudinal components of the electric and the magnetic fields, even when no charge as well as no »external» currents are assumed to exist and a »simple» medium is considered.

Formula (B.42) implies that the real part of k is positive ($\omega > 0$). Therefore the function $e^{i(\omega t-kz)}$ represents a field in which the surfaces of constant phase propagate in the direction of the positive z-axis, while $e^{i(\omega t+kz)}$ is associated with propagation in the negative z-direction. It follows from formulae (B.41) and (B.42) that the imaginary part of k is negative if the medium is conducting. This means that both waves, $e^{i(\omega t-kz)}$ and $e^{i(\omega t+kz)}$, are attenuated during the propagation in a conducting medium. If $\sigma = 0$, k is real and no attenuation occurs. From equations (B.48) and (B.49) it is seen that the conductivity of the medium causes a phase difference between the electric field and the magnetic field in both waves.

B.7. Boundary conditions

The scalar parameters ϵ, μ and σ characterizing a medium have above been considered constant in time and space (excluding the discussion at the end of Section B.4). The medium is then isotropic, linear and homogeneous. In a more general situation these parameters vary from point to point. Then the medium is not homogeneous. As is evident from the discussion of Section B.1, it is possible to use Maxwell's equations if these variations and their gradients are continuous and if the sources are also otherwise continuous. In the treatment of electromagnetic problems it is also necessary to know how electromagnetic vectors behave when moving across a surface which separates one medium from another. On a macroscopic scale the values of ϵ, μ and σ may change discontinuously at such a surface. Therefore it can be expected that the field vectors also have discontinuities at the surface.

Let us now deal generally with any surface of discontinuity occurring in an electromagnetic problem. The boundary conditions of the electromagnetic field vectors at this surface can be derived by first considering a thin transition layer where the values of the quantities on one side vary rapidly but continuously to the values on the other side. Assume a very small right circular cylinder to be placed in this layer such that its ends lie at the surfaces of the layer.

By applying Gauss' theorem and equation (B.10)

$$\oint_S \bar{D} \cdot \hat{n}\, da = \int_V \nabla \cdot \bar{D}\, dV = \int_V \rho_{true}\, dV = q_{true}, \tag{B.50}$$

where S is a closed surface which encloses the volume V, $\hat{n}$ is a unit vector pointing outwards at right angles to S and q_{true} is the total true charge within V.

If S is the above-mentioned small cylinder, formula (B.50) leads to the equation

$$(\bar{D}_2 \cdot \hat{n}_{12} - \bar{D}_1 \cdot \hat{n}_{12})\, \Delta a + \text{contribution from the side of the cylinder} = q_{true}. \tag{B.51}$$

Here $\hat{n}_{12}$ denotes the unit normal vector at the boundary from medium 1 to medium 2, $\bar{D}_2$ is the value of $\bar{D}$ at the boundary in medium 2, $\bar{D}_1$ is the corresponding value in medium 1 and Δa is the area of the end of the cylinder. Letting the thickness of the transition layer, *i.e.* the height of the cylinder, approach (macroscopically) zero and expressing q_{true} as $\delta_{true}\ \Delta a$ where δ_{true} is the true surface charge density at the boundary we obtain:

$$\hat{n}_{12} \cdot (\bar{D}_2 - \bar{D}_1) = \delta_{true}. \tag{B.52}$$

An assumption that $\bar{D}$ is finite was utilized to make the contribution from the side of the cylinder vanish in the limit. Equation (B.52) indicates that the discontinuity of the normal component of the $\bar{D}$ field at a boundary between two media is equal to the true surface charge density at the boundary surface.

Similarly from equation (B.11)

$$\hat{n}_{12} \cdot (\bar{B}_2 - \bar{B}_1) = 0 \tag{B.53}$$

so that the normal component of $\bar{B}$ is continuous.

Let us now replace the small cylinder in the transition layer with a rectangle whose plane is perpendicular to the surfaces of the layer and whose longer sides lie at the surfaces of the layer.

Stokes' theorem and equation (B.12) give

$$\oint_L \bar{E} \cdot d\bar{l} = \int_A \nabla \times \bar{E} \cdot \hat{n}\, da = -\int_A \frac{\partial \bar{B}}{\partial t} \cdot \hat{n}\, da \tag{B.54}$$

where L is a closed loop in space and A any surface bounded by L and the direction of the unit normal vector $\hat{n}$ is such that the rotation around L is positive, *i.e.* right-handed.

Assuming that L is the above-mentioned rectangle:

$$(\bar{E}_1 \cdot \hat{t} - \bar{E}_2 \cdot \hat{t})\Delta s + \text{contribution from the ends of the rectangle} = -\frac{\partial \bar{B}}{\partial t} \cdot \hat{n}\, \Delta s\, \Delta l. \tag{B.55}$$

In this equation $\hat{t}$ is a unit vector parallel to the surfaces of the transition layer and parallel to the side Δs of the rectangle. The vectors $\bar{E}_1$ and $\bar{E}_2$ are defined as $\bar{D}_1$ and $\bar{D}_2$ above, $\hat{n}$ is a unit vector normal to the rectangle and Δl is the length of the ends of the rectangle, *i.e.* the thickness of the transition layer. Let us again shrink Δl to zero. The contribution from the ends then vanishes because $\bar{E}$ is finite. Owing to an assumption that the derivative $\partial \bar{B}/\partial t$ remains finite, the right-hand side of formula (B.55) becomes zero. Expressing $\hat{t}$ as $\hat{t} = \hat{n}_{12} \times \hat{n}$ we thus obtain:

$$\hat{n} \cdot \hat{n}_{12} \times (\bar{E}_2 - \bar{E}_1) = 0, \tag{B.56}$$

but because the orientation of the rectangle, *i.e.* the direction of $\hat{n}$, is two-dimensionally arbitrary, the following equation has to be satisfied:

$$\hat{n}_{12} \times (\bar{E}_2 - \bar{E}_1) = 0. \tag{B.57}$$

This means that the tangential component of the vector $\bar{E}$ is continuous across the surface of discontinuity.

In the same way from equation (B.13):

$$\hat{n}_{12} \times (\bar{H}_2 - \bar{H}_1) = \bar{K}_{true} \tag{B.58}$$

where $\bar{K}_{true}$ is the true surface current density at the boundary. The derivative $\partial\bar{D}/\partial t$ was here assumed to remain finite, like $\partial\bar{B}/\partial t$ above. A non-zero value of $\bar{K}_{true}$ means that $\bar{j}_{true}$ is infinite at the boundary. So if a boundary on both sides of which the conductivities are finite is considered, $\bar{K}_{true}$ must be zero and then the tangential component of $\bar{H}$ is continuous:

$$\hat{n}_{12} \times (\bar{H}_2 - \bar{H}_1) = 0. \tag{B.59}$$

From equations (B.1) and (B.4) we could still derive new boundary conditions, for the normal component of $\bar{E}$ and for the tangential component of $\bar{B}$, respectively. Equations (B.8) and (B.9) would then give boundary conditions for $\bar{P}$ and $\bar{M}$ (*cf.* below). Using equations (B.21) and (B.22) the boundary condition for $\bar{D}$ can be expressed in terms of $\bar{E}$ and that for $\bar{H}$ in terms of $\bar{B}$.

The equation of continuity (B.18) is similar to Maxwell's equation (B.10), which leads to the boundary condition (B.52). Hence we may analogously deduce that

$$\hat{n}_{12} \cdot (\bar{j}_2 - \bar{j}_1) = -\frac{\partial \delta}{\partial t}. \tag{B.60}$$

Here δ is the total surface charge density. However, equation (B.60) is absolutely valid only in the absence of surface current. The presence of any type of surface current may give an additional term to the equation. Such a term is mathematically the result of the possible non-vanishing of the contribution from the side of the cylinder when treating the equation of continuity in the same way as equation (B.10) was treated in the derivation of formula (B.52). The only types of surface current are true $\bar{K}_{true}$ and magnetization $\bar{K}_M$:

$$\bar{K}_M = \hat{n}_{12} \times (\bar{M}_2 - \bar{M}_1). \tag{B.61}$$

The existence of polarization surface current would presume $\partial\bar{P}/\partial t$ to be infinite, which would presume that either $\partial\bar{E}/\partial t$ or $\partial\bar{D}/\partial t$ or both were infinite. As in the case of space charge the only types of surface charge are true δ_{true} and polarization δ_P:

$$\delta_P = -\hat{n}_{12} \cdot (\bar{P}_2 - \bar{P}_1). \tag{B.62}$$

Formulae (B.61) and (B.62) are boundary conditions of $\bar{M}$ and $\bar{P}$ (*cf.* above).

It was stated in Section B.3 that the equation of continuity (B.18) is valid for all types of charges and currents separately. Equation (B.60) is also applicable to all types of currents and surface charges separately provided the possible contribution of the corresponding surface current is taken into account.

Let us now consider a plane boundary and assume a Cartesian coordinate system whose z-axis is perpendicular to this plane. According to equation (B.57) the electric field components E_x and E_y are continuous at the plane $z = z_0$ in question for all values of x and y and at every time, *i.e.*

$$E_{1x}(x,y,z_0,t) = E_{2x}(x,y,z_0,t) \tag{B.63}$$

and

$$E_{1y}(x,y,z_0,t) = E_{2y}(x,y,z_0,t) \tag{B.64}$$

where x, y and t are arbitrary. Taking the derivative $\partial/\partial y$ of equation (B.63) and the derivative $\partial/\partial x$ of equation (B.64) and then subtracting yields

$$\frac{\partial E_{1x}}{\partial y} - \frac{\partial E_{1y}}{\partial x} = \frac{\partial E_{2x}}{\partial y} - \frac{\partial E_{2y}}{\partial x} \tag{B.65}$$

for all x, y and t at $z = z_0$. The left and right-hand sides of formula (B.65) are $-(\nabla \times \bar{E}_1)_z$ and $-(\nabla \times \bar{E}_2)_z$ at $z = z_0$, respectively. Hence using Maxwell's equation (B.3) we obtain:

$$\frac{\partial B_{1z}}{\partial t} = \frac{\partial B_{2z}}{\partial t} \tag{B.66}$$

for all x, y and t at $z = z_0$. In other words, it follows from the continuity of the tangential component of the electric field that the difference in the normal components of the magnetic field on each side of the boundary is constant in time. If it can be further assumed that this difference has vanished or will vanish at some time the continuity of the normal component of the magnetic field stated by formula (B.53) is obtained. At this point it can be pointed out that equation $\nabla \cdot \frac{\partial \bar{B}}{\partial t} = 0$, which leads to formula (B.66), is a direct consequence of Maxwell's equation (B.12) used in the derivation of formula (B.57) (*cf.* the end of Section B.1).

For harmonic time-dependence ($e^{i\omega t}$, $\omega \neq 0$) equation (B.66) involves the continuity of the normal component of the magnetic field. Hence, when considering time-harmonic fields the explicit use of the boundary condition (B.53) does not give any new information if the boundary condition (B.57) is also utilized (*cf.* the comment on equation (B.36) following from equation (B.37) in Section B.5).

The relationship between the continuities of the tangential component of $\bar{E}$ and of the normal component of $\partial\bar{B}/\partial t$ and of $\bar{B}$ was established above in the case of plane boundaries. However, since any surface can be approximated by a plane in a small neighbourhood of its any sufficiently regular point, it is evident that the conclusions of the relationship are not limited to the case of plane boundaries.

Similarly the boundary condition (B.58) concerning the tangential component of $\bar{H}$ implies that the discontinuity of the normal component of the quantity $\bar{j}_{true} + \partial\bar{D}/\partial t$ (see equation (B.13)) is equal to the negative divergence of $\bar{K}_{true}$. Using formula (B.52), equation (B.60) is then obtained for true charge and true current, complete with the comment about true surface current mentioned above. In analogy to Section B.3, an »inverse» relationship, which leads to the time derivative of equation (B.52), could also be presented. It was pointed out above that still more boundary conditions could be derived from Maxwell's equations. Thus the discussion of the relationships between the boundary conditions could also be continued.

B.8. Solution of an electromagnetic field from its sources

In principle it is possible to solve the fields $\bar{E}(\bar{r}, t)$ and $\bar{B}(\bar{r}, t)$ from Maxwell's equations (B.1)–(B.4) if the sources $\rho(\bar{r}, t)$ and $\bar{j}(\bar{r}, t)$ are known. This can be done using a vector potential $\bar{A}(\bar{r}, t)$ and a scalar potential $\phi(\bar{r}, t)$.

It follows from equation (B.2) that

$$\bar{B} = \nabla \times \bar{A} \qquad \text{(B.67)}$$

where $\bar{A}$ is some vector function, called the vector potential. Substitution of equation (B.67) in equation (B.3) gives

$$\nabla \times \left(\bar{E} + \frac{\partial\bar{A}}{\partial t}\right) = 0, \qquad \text{(B.68)}$$

so that

$$\bar{E} = -\nabla\phi - \frac{\partial\bar{A}}{\partial t} \qquad \text{(B.69)}$$

where ϕ is some scalar function, called the scalar potential. From equations (B.1), (B.4), (B.67) and (B.69) we obtain:

$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla \cdot \bar{A}) = -\frac{\rho}{\epsilon_0} \tag{B.70}$$

and

$$\nabla^2\bar{A} - \nabla\left(\nabla \cdot \bar{A} + \mu_0\epsilon_0 \frac{\partial\phi}{\partial t}\right) - \mu_0\epsilon_0 \frac{\partial^2\bar{A}}{\partial t^2} = -\mu_0\bar{j}. \tag{B.71}$$

Equations (B.70) and (B.71) look rather complicated. However, the fields $\bar{E}$ and $\bar{B}$ and hence the physics remain unchanged if $\bar{A}$ is changed to

$$\bar{A}' = \bar{A} + \nabla\psi \tag{B.72}$$

and ϕ is changed to

$$\phi' = \phi - \frac{\partial\psi}{\partial t}, \tag{B.73}$$

where $\psi(\bar{r}, t)$ is an arbitrary scalar function. Now $\psi(\bar{r}, t)$ can be chosen so that

$$\nabla \cdot \bar{A}' + \mu_0\epsilon_0 \frac{\partial\phi'}{\partial t} = 0, \tag{B.74}$$

i.e.

$$\nabla^2\psi - \mu_0\epsilon_0 \frac{\partial^2\psi}{\partial t^2} = -\nabla \cdot \bar{A} - \mu_0\epsilon_0 \frac{\partial\phi}{\partial t}. \tag{B.75}$$

The transformation included in equations (B.72) and (B.73) is known as a gauge transformation. Choosing the divergence of the vector potential is called choosing a gauge. If the relation (B.74) (the so-called Lorentz condition) is valid, the Lorentz gauge is involved.

Let us now assume that the gauge transformation has already been made and the vector potential $\bar{A}$ and the scalar potential ϕ in equations (B.70) and (B.71) satisfy the Lorentz condition (B.74). Then

$$\nabla^2\phi - \mu_0\epsilon_0 \frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{B.76}$$

and

$$\nabla^2\bar{A} - \mu_0\epsilon_0 \frac{\partial^2\bar{A}}{\partial t^2} = -\mu_0\bar{j}. \tag{B.77}$$

It is seen that both potentials satisfy exactly similar equations. The solutions of equations (B.76) and (B.77) are

$$\phi(\bar{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(\bar{r}', t - \frac{R}{c}\right)}{R} dV' \tag{B.78}$$

and

$$\bar{A}(\bar{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\bar{j}\left(\bar{r}', t - \frac{R}{c}\right)}{R} dV' \tag{B.79}$$

where $R = |\bar{r} - \bar{r}'|$ and, as above, $c = (\mu_0\epsilon_0)^{-1/2}$ is the speed of light. The integration volume must include all points $\bar{r}'$ in space where the source function in the integrand differs from zero to obtain a complete solution. The time of the source fields is retarded from the time of observation by an amount equal to the time taken by light to travel from the source point to the point of observation. Therefore the potentials (B.78) and (B.79) are called retarded potentials. Mathematically, advanced potentials where the time of the source fields is $t + R/c$ are also acceptable. However, they seem to violate causality and are excluded. By direct calculation with the equation of continuity (B.18) it can be shown that equations (B.78) and (B.79) satisfy the Lorentz condition (B.74) provided the integral $\int_S da'\hat{n}'\cdot \bar{j}(\bar{r}', t - (R/c))/R$ vanishes as S goes to infinity.

Using equations (B.67), (B.69), (B.78) and (B.79) the electric field and the magnetic field are obtained:

$$\bar{E}(\bar{r}, t) = \frac{1}{4\pi\epsilon_0}\left(\int \frac{[\rho]\bar{R}}{R^3} dV' + \frac{1}{c}\int \frac{[\dot{\rho}]\bar{R}}{R^2} dV' - \frac{1}{c^2}\int \frac{[\dot{\bar{j}}]}{R} dV'\right) \tag{B.80}$$

and

$$\bar{B}(\bar{r}, t) = \frac{\mu_0}{4\pi}\left(\int \frac{[\bar{j}]\times\bar{R}}{R^3} dV' + \frac{1}{c}\int \frac{[\dot{\bar{j}}]\times\bar{R}}{R^2} dV'\right). \tag{B.81}$$

The dot above a symbol means the time derivative, the square brackets denote retardation and $\bar{R} = \bar{r} - \bar{r}'$. If the surface integral $\int_S da'\hat{n}' \cdot [\bar{j}]\bar{R}/R^2$ vanishes when S goes to infinity the expression of $\bar{E}(\bar{r}, t)$ can be developed further:

$$\bar{E}(\bar{r}, t) = \frac{1}{4\pi\epsilon_0}\left(\int \frac{[\rho]\bar{R}}{R^3} dV' + \frac{1}{c}\int\left(\frac{2[\bar{j}]\cdot\bar{R}\bar{R}}{R^4} - \frac{[\bar{j}]}{R^2}\right)dV' + \frac{1}{c^2}\int \frac{([\dot{\bar{j}}]\times\bar{R})\times\bar{R}}{R^3} dV'\right). \tag{B.82}$$

The equation of continuity (B.18) is also needed in the derivation of equation (B.82) from formula (B.80). The last terms of equations (B.81) and (B.82) represent the so-called radiation fields.

The above formulae of this section can be extended to the case where Maxwell's equations (B.23), (B.2), (B.3) and (B.24) are satisfied with time- and space-constants ϵ and μ; $\rho(\bar{r}, t)$, $\bar{j}(\bar{r}, t)$, μ_0, ϵ_0 and $c = (\mu_0\epsilon_0)^{-1/2}$ have simply to be replaced by $\rho_{true}(\bar{r}, t)$, $\bar{j}_{true}(\bar{r}, t)$, μ, ϵ and $\nu = (\mu\epsilon)^{-1/2}$. If $\bar{j}_{true}$ now depends on $\bar{E}$, as for example in Ohm's law (B.25), equations (B.80) and (B.82) are integral-differential equations for the electric field, and the situation is complicated.

Let us, however, treat the case where $\bar{j}_{true}$ can be expressed as $\bar{j}_{true} = \sigma\bar{E} + \bar{j}'_{true}$ in a different manner. The current $\bar{j}'_{true}$ is considered an »external» current. The introduction of a vector potential $\bar{A}(\bar{r}, t)$ and a scalar potential $\phi(\bar{r}, t)$ according to formulae (B.67) and (B.69) satisfies Maxwell's equations (B.2) and (B.3). Equation (B.23) then yields formula (B.70) with ρ/ϵ_0 replaced by ρ_{true}/ϵ. The equation corresponding to formula (B.71) can be written as

$$\nabla^2\bar{A} - \nabla\left(\nabla\cdot\bar{A} + \mu\rho\phi + \mu\epsilon\frac{\partial\phi}{\partial t}\right) - \mu\sigma\frac{\partial\bar{A}}{\partial t} - \mu\epsilon\frac{\partial^2\bar{A}}{\partial t^2} = -\mu\bar{j}'_{true}\,. \tag{B.83}$$

This equation suggests that the »reasonable» Lorentz condition is now

$$\nabla\cdot\bar{A} + \mu\sigma\phi + \mu\epsilon\frac{\partial\phi}{\partial t} = 0\,. \tag{B.84}$$

This equation is made valid with a gauge transformation (B.72) and (B.73) where ψ satisfies the equation

$$\nabla^2\psi - \mu\sigma\frac{\partial\psi}{\partial t} - \mu\epsilon\frac{\partial^2\psi}{\partial t} = -\nabla\cdot\bar{A} - \mu\sigma\phi - \mu\epsilon\frac{\partial\phi}{\partial t} \tag{B.85}$$

(*cf.* formulae (B.74) and (B.75)). In analogy to the discussion above we assume that the Lorentz condition (B.84) is satisfied by $\bar{A}$ and ϕ. From equations (B.70) with ρ_{true}/ϵ and (B.83) it follows that

$$\nabla^2\phi - \mu\sigma\frac{\partial\phi}{\partial t} - \mu\epsilon\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho_{true}}{\epsilon} \tag{B.86}$$

and

$$\nabla^2\bar{A} - \mu\sigma\frac{\partial\bar{A}}{\partial t} - \mu\epsilon\frac{\partial^2\bar{A}}{\partial t^2} = -\mu\bar{j}'_{true}\,. \tag{B.87}$$

Both ϕ and $\bar{A}$ again have exactly similar equations, but it should be noted that the right-hand side of formula (B.86) involves the total true charge, while $\bar{j}'_{true}$ appearing in the right-hand side of equation (B.87) merely designates the »external» true current.

Equations (B.86) and (B.87), which also contain the first time derivatives of the potentials, do not in the general case have solutions as simple as formulae (B.78) and (B.79) are for equations (B.76) and (B.77). Let us, however, consider a harmonic time-dependence $e^{i\omega t}$ ($\omega \geqslant 0$). Then using formula (B.41) equations (B.84), (B.86) and (B.87) can be expressed as

$$\nabla \cdot \bar{A} + \frac{ik^2}{\omega}\phi = 0\,, \tag{B.88}$$

$$\nabla^2\phi + k^2\phi = -\frac{\rho_{true}}{\epsilon} \tag{B.89}$$

and

$$\nabla^2\bar{A} + k^2\bar{A} = -\mu\bar{j}'_{true}\,. \tag{B.90}$$

But with harmonic time-dependence equations (B.74), (B.76) and (B.77) connected with free space can be written in exactly the same form, utilizing the propagation constant k_0 of free space:

$$k_0 = \omega\sqrt{\mu_0\epsilon_0} = \frac{\omega}{c}\,, \tag{B.91}$$

which is real and non-negative. Hence, obviously, the results concerning free space are also formally applicable to Maxwell's equations (B.23), (B.2), (B.3) and (B.24) with $\bar{j}_{true} = \sigma\bar{E} + \bar{j}'_{true}$ and with a harmonic time-dependence; ϵ_0 will be replaced by ϵ, μ_0 by μ, c by ω/k, ρ by ρ_{true} and $\bar{j}$ by $\bar{j}'_{true}$ (see also formula (B.42)).

The discussion of this section implicitly presumes that Maxwell's equations are valid everywhere in three-dimensional space and at every time involved. Hence, according to Section B.1, the sources $\rho(\bar{r}, t)$ and $\bar{j}(\bar{r}, t)$ should be continuous and finite. Thus idealized abrupt changes are in principle not permissible. They can, however, be regarded as limits of rapid but continuous variations, for which the above discussion is valid. This implies that formulae (B.78)–(B.82) containing integrals are obviously applicable for such idealized discontinuous sources. If Dirac delta functions are involved in the expressions of the sources, infinities appear in addition to discontinuities. Let us, however, refer to Section B.1 and assume that the use of equations (B.78)–(B.82) is also permissible then.

B.9. Poynting vector

From Maxwell's equations (B.3) and (B.4) it follows that

$$\nabla \cdot \left(\bar{E} \times \frac{1}{\mu_0} \bar{B}\right) = -\bar{E} \cdot \bar{j} - \epsilon_0 \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} - \frac{1}{\mu_0} \bar{B} \cdot \frac{\partial \bar{B}}{\partial t} \,. \tag{B.92}$$

Let us integrate this equation over a volume V which is enclosed by a closed surface S. Gauss' theorem then gives

$$\oint_S \left(\bar{E} \times \frac{1}{\mu_0} \bar{B}\right) \cdot \hat{n} da + \int_V \bar{E} \cdot \bar{j} dV = -\int_V \left(\epsilon_0 \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} + \frac{1}{\mu_0} \bar{B} \cdot \frac{\partial \bar{B}}{\partial t}\right) dV \,. \tag{B.93}$$

This equation is called Poynting's theorem and it can be interpreted as follows: The right-hand side represents the rate of decrease of the electromagnetic field energy within V. The second term of the left-hand side is the work per unit time done by the field in V (*cf.* Section B.2). The first term on the left-hand side is the flow of electromagnetic energy out of V through S. The vector

$$\bar{N} = \frac{1}{\mu_0} \bar{E} \times \bar{B} \tag{B.94}$$

is called the Poynting vector. Poynting's theorem may also be written in a slightly different form which can be derived from equations (B.12) and (B.13). Then the Poynting vector is defined as $\bar{E} \times \bar{H}$.

The Poynting vector was introduced as an integrand in equation (B.93). This equation does not change when an arbitrary vector whose divergence vanishes is added to the Poynting vector. In other words, the surface integral of the Poynting vector over a closed surface is the only thing that has significance. However, on the grounds that the Poynting vector is also related to the momentum and angular momentum of the electromagnetic field, evidence is obtained that the Poynting vector really gives the space distribution of energy flow in an electromagnetic field (FURRY, 1969, ROMER, 1966 and 1967).

Since the Poynting vector is the product of two physical quantities, the real parts must first be taken before multiplication if the electromagnetic field is mathematically described with complex quantities (see Section A.1). However, it is easy to show that the time average over the period of the product of two physical quantities oscillating harmonically with time (with $\omega \neq 0$) is equal to half of the real part of the product of one complex quantity and the complex conjugate of the other. Hence the time average of the Poynting vector of an electromagnetic field with a harmonic time-dependence is

$$\langle \bar{N} \rangle = Re \bar{N}_c \tag{B.95}$$

where $\bar{N}_c$ is the so-called complex Poynting vector defined by

$$\bar{N}_c = \frac{1}{2\mu_0} \bar{E} \times \bar{B}^\star . \tag{B.96}$$

As an example, let us consider the Poynting vector of the electromagnetic plane wave given by equations (B.48) and (B.49) with $\bar{E}_1 = 0$. Then

$$\bar{N}_c = \frac{k^\star}{2\mu_0\omega} \bar{E}_0 \times (\hat{e}_z \times \bar{E}_0^\star) e^{-i(k-k^\star)z} = \frac{k^\star \bar{E}_0 \cdot \bar{E}_0^\star}{2\mu_0\omega} e^{2k_2 z} \hat{e}_z \tag{B.97}$$

where k_2 denotes the imaginary part of k and the fact that $\bar{E}_0$ is perpendicular to $\hat{e}_z$ has been utilized. Equations (B.95) and (B.97) imply that

$$\langle N \rangle = \frac{k_1 \bar{E}_0 \cdot \bar{E}_0^\star}{2\mu_0\omega} e^{2k_2 z} \hat{e}_z \tag{B.98}$$

in which the real part of k has been denoted by k_1. Because k_1 is positive, the evident result that the energy flows on average in the direction of the phase propagation is seen from formula (B.98). The other possible Poynting vector, *i.e.* $\bar{E} \times \bar{H}$, would lead to a time average equal to $\langle \bar{N} \rangle$ given by equation (B.98) multiplied by a positive factor μ_0/μ, and so the conclusion of the direction of the energy flow would not change. The directions of phase propagation and of the Poynting vector need not be equal, and it is the latter that is more important.

B.10. Principle of superposition

A very important principle in the discussion of electromagnetic fields is the so-called principle of (linear) superposition. It states that the electromagnetic field caused by several sources is the sum of the fields produced by each individual source separately. Here the source is any charge and current distribution satisfying the equation of continuity (B.18). The linearity of Maxwell's equations with respect to the fields and sources is consistent with the principle of superposition.

Appendix C. On cylindrical electromagnetic fields

C.1. Poynting vector of a transverse magnetic cylindrical electromagnetic field

Let us consider an electromagnetic field whose time-dependence is harmonic ($e^{i\omega t}$, $\omega > 0$) in a »simple» medium characterized by a permittivity ϵ, a permeability μ and a conductivity σ. We assume that no »external» currents exist. The field then satisfies Maxwell's equations (B.35)–(B.38). Let us use cylindrical coordinates ρ, φ and z and assume that the field is independent of φ and the z-dependence is given by e^{-iqz} where q is a complex constant different from the propagation constant k of the medium (formulae (B.41) and (B.42)).

Equations (B.37) and (B.38) can be written in component form as formulae (4.11)–(4.16) with J equal to zero and $i\omega\mu_0\epsilon_0$ replaced by $\mu(\sigma + i\omega\epsilon)$. Hence the components E_ρ, E_z and B_φ are independent of the components E_φ, B_ρ and B_z. We now assume that the latter are zero. Thus the magnetic field has only a φ-component, which implies that the case is transverse magnetic (TM).

It follows from the wave equation (B.43) that E_z satisfies equation (4.7) with the right-hand side equal to zero and k_0 in the definition of η replaced by k (formulae (4.8) and (4.9)). The equation is Bessel's differential equation of the zeroth order (A.20) and so the general solution for E_z can be expressed as

$$E_z(\rho, z, t) = (FH_0^{(1)}(\eta\rho) + DH_0^{(2)}(\eta\rho))e^{i(\omega t - qz)} . \tag{C.1}$$

The components E_ρ and B_φ are obtained using formulae (4.17) and (4.18) and substituting $-i\mu(\sigma + i\omega\epsilon)$ for $\omega\mu_0\epsilon_0$:

$$E_\rho(\rho, z, t) = \frac{iq}{\eta} (FH_1^{(1)}(\eta\rho) + DH_1^{(2)}(\eta\rho))e^{i(\omega t - qz)} \tag{C.2}$$

and

$$B_\varphi = \frac{\mu(\sigma + i\omega\epsilon)}{\eta} (FH_1^{(1)}(\eta\rho) + DH_1^{(2)}(\eta\rho))e^{i(\omega t - qz)} . \tag{C.3}$$

A direct substitution shows that this electromagnetic field really satisfies Maxwell's equations (B.35)–(B.38), the first two of which are consequences of the two latter.

Using equations (B.41) and (C.1)–(C.3) the complex Poynting vector (B.96) has the expression

$$\bar{N}_c = \frac{1}{2\mu_0} \cdot (-E_z B_\varphi^\star \hat{e}_\rho + E_\rho B_\varphi \hat{e}_z) \tag{C.4}$$

$$= \frac{|e^{-iqz}|^2}{2\omega\mu_0|\eta|^2}[ik^{\star 2}\eta(|F|^2 H_0^{(1)}(\eta\rho)H_1^{(1)\star}(\eta\rho) + |D|^2 H_0^{(2)}(\eta\rho)H_1^{(2)\star}(\eta\rho)$$

$$+ F^\star D H_0^{(2)}(\eta\rho)H_1^{(1)\star}(\eta\rho) + FD^\star H_0^{(1)}(\eta\rho)H_1^{(2)\star}(\eta\rho))\hat{e}_\rho$$

$$+ k^{\star 2} q(|F|^2\,|H_1^{(1)}(\eta\rho)|^2 + |D|^2\,|H_1^{(2)}(\eta\rho)|^2$$

$$+ F^\star D H_1^{(1)\star}(\eta\rho)H_1^{(2)}(\eta\rho) + FD^\star H_1^{(1)}(\eta\rho)H_1^{(2)\star}(\eta\rho))\,\hat{e}_z].$$

Let us assume that ρ approaches infinity and substitute the asymptotic expressions (A.58) and (A.59) in equation (C.4). Then

$$\bar{N}_c \approx \frac{e^{2q_2 z}}{\pi\omega\mu_0|\eta|^3\rho}[k^{\star 2}\eta(-|F|^2 e^{-2\eta_2\rho} + |D|^2 e^{2\eta_2\rho} \tag{C.5}$$

$$- i(F^\star D e^{-i2\eta_1\rho} + FD^\star e^{i2\eta_1\rho}))\hat{e}_\rho + k^{\star 2} q(|F|^2 e^{-2\eta_2\rho}$$

$$+ |D|^2 e^{2\eta_2\rho} + i(-F^\star D e^{-i2\eta_1\rho} + FD^\star e^{i2\eta_1\rho}))\hat{e}_z]$$

where η_1 and η_2 are the real and imaginary parts, respectively, of η, and Imq is denoted by q_2.

If D equals zero and $F \neq 0$, it is obtained that

$$\bar{N}_c = \frac{|F|^2 e^{2q_2 z}}{2\omega\mu_0|\eta|^2}(ik^{\star 2}\eta H_0^{(1)}(\eta\rho)H_1^{(1)\star}(\eta\rho)\hat{e}_\rho + k^{\star 2} q\,|H_1^{(1)}(\eta\rho)|^2\hat{e}_z) \tag{C.6}$$

$$\approx \alpha_1(\rho, z)k^{\star 2}(-\eta\hat{e}_\rho + q\hat{e}_z)$$

where the real and positive function $|F|^2 e^{-2\eta_2\rho} e^{2q_2 z}/\pi\omega\mu_0|\eta|^3\rho$ is denoted by $\alpha_1(\rho, z)$. Using equation (B.95) and writing $q_1 = Req$, $\beta_1 = Rek^2$ and $\beta_2 = Imk^2$, formula (C.6) gives the following expression for the time average of the energy flow:

$$\langle\bar{N}\rangle \approx \alpha_1(\rho, z)(\beta_1(-\eta_1\hat{e}_\rho + q_1\hat{e}_z) + \beta_2(-\eta_2\hat{e}_\rho + q_2\hat{e}_z))\,. \tag{C.7}$$

Similarly in the case $F = 0$ and $D \neq 0$ it follows that

$$\langle\bar{N}\rangle \approx \alpha_2(\rho, z)(\beta_1(\eta_1\hat{e}_\rho + q_1\hat{e}_z) + \beta_2(\eta_2\hat{e}_\rho + q_2\hat{e}_z)) \tag{C.8}$$

where the real and positive function $|D|^2 e^{2\eta_2\rho} e^{2q_2 z}/\pi\omega\mu_0|\eta|^3\rho$ is denoted by

$\alpha_2(\rho, z)$. If $0 < \arg \eta \leqslant \pi/2$, $\alpha_1(\rho, z)$ and $\alpha_2(\rho, z)$ exponentially approach zero and infinity, respectively, as ρ approaches infinity. If $-\pi/2 < \arg\eta < 0$, the limits are conversely.

By considering equations (C.1)–(C.3) and the asymptotic expressions (A.58) and (A.59) of the Hankel functions it can be seen that in both cases, $D = 0$ and $F = 0$, the vectors multiplied by β_1 and β_2 in equations (C.7) and (C.8) give the direction of the propagation of constant phase and the direction of the exponential increase of the amplitude, respectively. In other words, whichever Hankel function is chosen to represent the ρ-dependence of the electromagnetic field in question, the direction of the time average of the Poynting vector is equal to the direction of the vector: Rek^2 times the direction of propagation of constant phase minus Imk^2 times the direction of exponential attenuation. These direction vectors are not unit vectors or equal in length but those appearing in formulae (C.7) and (C.8). According to equation (B.41) Rek^2 is positive and Imk^2 is non-positive.

In the special case of a non-conducting medium Imk^2 is zero and then the directions of the average energy flow and of the propagation of constant phase coincide. It follows from the defining equation of η (4.8) that if Imk^2 equals zero, then η and q satisfy the equation $\eta_1\eta_2 = -q_1q_2$, and hence the vectors $\mp\eta_1\hat{e}_\rho + q_1\hat{e}_z$ and $\pm\eta_2\hat{e}_\rho - q_2\hat{e}_z$, *i.e.* the directions of propagation and attenuation, are orthogonal in both cases $D = 0$ and $F = 0$ (*cf.* Barlow and Cullen, 1953, p. 336).

If both phase propagation and exponential attenuation with respect to a space coordinate (ρ or z) take place in the same direction, the energy flows on average in the same direction with respect to this coordinate. If no phase propagation or no exponential attenuation occurs in a space coordinate, the energy flows in the direction of the only occurring. However, in a non-conducting medium the vanishing of phase propagation with respect to a space coordinate makes the energy flow vanish in this direction. If both phase propagation and exponential attenuation with respect to a space coordinate vanish, no energy flow with respect to this coordinate takes place.

Using equation (C.5) the energy flow for any linear combination of the Hankel functions of the first and second kind can be studied. For example, if the medium is non-conducting and F and D are equal, we obtain:

$$\langle \bar{N} \rangle \approx \frac{2|F|^2 e^{2q_2z} k^2}{\pi\omega\mu_0|\eta|^3\rho} ((\eta_1 \sinh 2\eta_2\rho + \eta_2 \cos 2\eta_1\rho)\hat{e}_\rho + (q_1 \cosh 2\eta_2\rho - q_1 \sin 2\eta_1\rho)\hat{e}_z). \tag{C.9}$$

Let us assume that η_2 is positive. Since ρ is large, equation (C.9) then reduces to

$$\langle \bar{N} \rangle \approx \alpha_2(\rho_1 z) k^2 (\eta_1 \hat{e}_\rho + q_1\hat{e}_z), \tag{C.10}$$

but this is exactly the expression obtained from equation (C.8) with $\beta_2 = 0$. The result simply establishes that it is the infinitely growing function $H_\nu^{(2)}(\eta\rho)$ that is of importance for large values of ρ.

The same results as regards the direction of the energy flow would also have been obtained in this section with the employment of the other commonly used Poynting vector $\bar{N}' = \bar{E} \times \bar{H}$ (*cf.* Section B.9).

C.2. Surface waves

It seems that the question of the directions of propagation and attenuation could be treated more thoroughly than usual in the literature. A basic condition for considering the propagation and attenuation of an electromagnetic field of the type in Sections 4.2 and 4.3 is that the complex half plane where the quantity corresponding to η of formulae (4.8) and (4.9) lies is clearly defined. (An example of the importance of the correct choice of the half-plane, *i.e.* the branch or the sign, of a square root is the famous »error in sign» in Sommerfeld's paper (SOMMERFELD, 1909), where the electromagnetic field caused by a vertical electric dipole at the interface between two half-spaces of different electromagnetic properties is discussed (WAIT, 1964, pp. 158–159; NORTON, 1935; NIESSEN, 1937; STRATTON, 1941, p. 585). Sommerfeld subsequently published another paper on the same subject and without the error (SOMMERFELD, 1926).)

Let us now briefly consider Stratton's treatment of cylindrical waves (STRATTON, 1941). On page 360 he establishes that the Hankel function of the first kind is related to a wave which phase-travels asymptotically radially outwards. He has not explicitly defined the half-plane of the quantity $\sqrt{k^2-h^2}$ appearing in the argument of the Hankel function and corresponding to the above-mentioned η, but taking into account formula (A.58) of this work and the time-dependence $e^{-i\omega t}$ used by Stratton, it is obvious that the real part of $\sqrt{k^2-h^2}$ is positive.

When discussing the propagation of electromagnetic fields along circular cylinders Stratton has chosen the Hankel function $H^{(1)}$ for the fields outside the cylinder »to ensure the proper behaviour at infinity» (STRATTON, 1941, p. 524). The nature of this »proper behaviour» is not explained. However, considering the numerical example on Stratton's pages 529–530, where the field propagates along a copper wire situated in the air, it can be shown that the argument of the Hankel function $H^{(1)}$ has a negative real and a positive imaginary part. Hence the corresponding field propagates asymptotically radially inwards, thus differing from the statement on page 360. This also conflicts with Ikrath's reasoning on the same topic (IKRATH, 1957, pp. 24–25).

The positive imaginary part of the argument makes the field damp exponentially outwards. So the field is confined to the neighbourhood of the surface of the cylinder along which it propagates. Such an electromagnetic field is a so-called surface wave (GOUBAU, 1950, BARLOW and CULLEN, 1953, SOMMERFELD, 1959, p. 156, WAIT, 1964, JONES, 1964, pp. 415–437).

An electromagnetic surface wave is defined as an electromagnetic field that propagates without outward radiation along an interface between two different media (BARLOW and CULLEN, 1953, p. 329). It is known from optics that there is a special angle of incidence, the so-called Brewster angle, for which there is no reflection for a wave polarized in the plane of incidence (*e.g.* STRATTON, 1941, pp. 497 and 516). Thus a polarized wave of this kind incident on a surface at the Brewster angle, which is complex in general, is evidently a surface wave (BARLOW and CULLEN, 1953, pp. 329 and 335, STRATTON, 1941, pp. 516–524). On their page 329 Barlow and Cullen discuss a discrepancy in the use of the word »surface wave», which is related to Sommerfeld's above-mentioned »error in sign».

The most important instance is where the external medium is non-conducting, for example to a good approximation air, which is also assumed here. Since the outside medium is loss free and outward radiation may not occur, all energy flow has to take place along the surface and inwards.

The case in which both media are infinite half-spaces and the interface hence a plane and no transverse space-dependence occurs is the easiest one to treat theoretically (ZENNECK, 1907, BARLOW and CULLEN, 1953, p. 330, SOMMERFELD, 1959, pp. 160–161). Propagation of a surface wave along an infinitely long circular cylinder, discussed *e.g.* by STRATTON, 1941, pp. 524–537, as mentioned above, also has a simple theoretical description (see GOUBAU, 1950, pp. 1119–1123, BARLOW and CULLEN, 1953, p. 331, SOMMERFELD, 1959, pp. 177–185): If no φ-dependence appears the transverse magnetic field inside the cylinder is expressed by equations (4.10), (4.19) and (4.20) with $J=0$ and with the electromagnetic parameters characterizing the material of the cylinder. The coefficient in formula (4.20) will then change to $[\mu(\sigma+i\omega\epsilon)/\eta]_{\text{cyl}}$. Outside the cylinder the transverse magnetic field is described by equations (C.1)–(C.3) with the parameters of the outside medium and with $F=0$ or $D=0$ such that the field is attenuated outwards. The latter requirement ensures that the field is confined to the vicinity of the surface. Since the medium is non-conducting, it further implies that the phase propagation occurs inwards, if q satisfies formula (4.2). Then the energy also flows inwards (see Section C.1). The boundary conditions at the surface of the cylinder constitute two linear homogeneous equations for the unknown coefficients, and in order to have a solution the determinant of this system of equations must vanish. This leads to an equation for the longitudinal propagation constant q, which is thus not arbitrary. The equation is transcendental, and q has no general explicit solution.

If a metal cylinder (or wire) situated in the air is considered, the transcendental determinant equation may be approximated and solved quite simply by iteration (STRATTON, 1941, pp. 528–539, SOMMERFELD, 1959, pp. 182–185). The approximation depends on the radius of the wire and on the frequency as seen in these authors' examples, which concern the so-called principal wave. With the numerical values of these examples the above-described solution shows that the longitudinal propagation constants lie in the fourth quadrant of the complex plane according to formula (4.2). In fact, in the mathematical calculations Stratton and Sommerfeld use the opposite sign in the exponent describing the longitudinal space-dependence and the time-dependence, to the discussion in this work. Therefore the longitudinal propagation constants presented by Sommerfeld and Stratton are situated in the first quadrant.

I have also studied theoretically transverse magnetic waves propagating along a plane or along a cylindrical surface and dealt mathematically with the case in which the probably unphysical solution is chosen to describe the transverse space-dependence of the electromagnetic field in the outside medium, *i.e.* the outward increasing function. It can be shown that with fixed parameters of the media the determinant equation allows only one solution for the outward perpendicular space dependence in the case of a plane surface. But in the case of a cylinder, the determinant equation may give solutions with both choices of the Hankel function outside the cylinder. I have shown this numerically for the values of the example of a copper wire presented by STRATTON, 1941, pp. 529–530, and by SOMMERFELD, 1952, pp. 183–184. The following statements are also based on calculations using these numerical values.

The value of the longitudinal propagation constant calculated from the determinant equation varies according to the choice of the Hankel function. In the case of the outward growing Hankel function, with the use of time-dependence $e^{-i\omega t}$ the same solution from the determinant equation is not obtained as with $e^{i\omega t}$, though these time-dependencies are physically identical. This contradiction, however, seems to be merely apparent, since the solution obtained with $e^{-i\omega t}$ exists in the case of $e^{i\omega t}$ in another branch of the Hankel function (see Section A.6). If the outward attenuating Hankel function is used, both choices of the time factor give the same result directly.

On the other hand, a mathematically acceptable solution to a problem may be quite unphysical. Hence the fact that the outward increasing case permits a mathematical solution does not remove or decrease the significance of the unphysicality caused by the infinite growth when moving outwards from the cylinder. However, let us still refer to the exponentially infinite growth in the $-z$-direction, which also seems unphysical. As pointed out in Section 4.2, such an idealization might cause other unphysicalities, too.

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