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DETERMINATION OF MOVEMENTS IN THE EARTH'S CRUST BY THE FINITE ELEMENT METHOD

by

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Abstract

A new, iterative method for the solution of the differential equations of slow, viscous, incompressible flow is presented. The procedure can be applied in connection with numerical methods. The equations needed in the finite element method are derived. Based on these equations a computer program dealing with plane and axi-symmetric cases has been developed. The program has been applied to the determination of movements and stresses in the earth's crust and mantle under the loading due to erosion and sedimentation. Some results from these calculations are presented.

1. *Introduction*

KAITERA [2], [3], [4], [5] has presented the hypothesis that the changes resulting from erosion and sedimentation have a considerable influence on the movements and stresses in the earth's crust and upper mantle. To give support to this theory, NISKANEN and KUTVONEN [2], [6], [7] have done calculations and obtained interesting results. In these calculations the crust and mantle have been assumed to behave like an incompressible viscous fluid. As the flow is very slow, the inertia effects could be omitted in the formulation of the problem. The applicability of the Fourier-series solution employed was restricted, however, to relatively simple geometry, which was composed of horizontal layers of constant thickness and viscosity.

In this paper a solution process based on an iteration scheme combined with the finite element method is presented. By this procedure problems with arbitrary geometry, loading and viscosity distributions can be solved. It can be applied to study the changes to previous results due to more realistic erosion-sedimentation and viscosity distributions and due to consideration [9] the spherical form of the earth. Some results determined by a computer program developed for plane and axi-symmetric cases are presented.

2. *Equations of creeping flow*

The governing equations of steady state, slow (creeping) flow of viscous, incompressible fluid [1] are given in this chapter. Rectangular right-handed Cartesian co-ordinates x, y, z and matrix notation are employed.

The strain rates of the flow are defined by means of the derivatives of the components v_x, v_y and v_z of the velocity vector $\vec{v}$ from expressions

$$\{d\} = [\partial]\,\{v\} \tag{1}$$

where

$$\{d\} = \begin{Bmatrix} d_x \\ d_y \\ d_z \\ d_{yz} \\ d_{zx} \\ d_{xy} \end{Bmatrix}, \quad [\partial] = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \\ \partial/\partial y & \partial/\partial x & 0 \end{bmatrix}, \quad \{v\} = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}. \tag{2}$$

The state of stress is defined by a column vector $\{\sigma\}$, composed of the stress components, by means of which the components T_x, T_y and T_z of stress vector $\vec{T}$ acting on any surface are obtained from equations

$$\{T\} = [n]^T\,\{\sigma\} \tag{3}$$

where

$$\{T\} = \begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix}, \quad [n] = \begin{bmatrix} n_x & 0 & 0 \\ 0 & n_y & 0 \\ 0 & 0 & n_z \\ 0 & n_z & n_y \\ n_z & 0 & n_x \\ n_y & n_x & 0 \end{bmatrix}, \quad \{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix} \tag{4}$$

and where n_x, n_y and n_z are the components of an outer unit normal vector $\vec{n}$ to the surface.

In creeping flow the inertia forces can be omitted and stresses $\{\sigma\}$ must hence satisfy the following equations of equilibrium

$$[\partial]^T\{\sigma\} + \{F\} = \{0\} \tag{5}$$

where

$$\{F\} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} \tag{6}$$

is a column vector composed of the components of the body force vector $\vec{F}$ per unit volume.

The velocity components $\{v\}$ have further to satisfy the continuity equation which is in case of incompressible flow

$$\{\nabla\}^T\{v\} = 0 \tag{7}$$

where

$$\{\nabla\} = \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix} . \tag{8}$$

When equation (7) is expressed in terms of the strain rates, the result

$$\mathrm{d} = 0 \tag{9}$$

is obtained for the rate of dilatation $\mathrm{d} \equiv \mathrm{d}_x + \mathrm{d}_y + \mathrm{d}_z$.

The relations between stresses and strain rates are linear for Newtonian fluids and can be expressed by equations

$$\begin{cases} \sigma_x = 2\mu \mathrm{d}_x + \lambda \mathrm{d} - p \\ \sigma_y = 2\mu \mathrm{d}_y + \lambda \mathrm{d} - p \\ \sigma_z = 2\mu \mathrm{d}_z + \lambda \mathrm{d} - p \\ \tau_{yz} = \mu \mathrm{d}_{yz} \\ \tau_{zx} = \mu \mathrm{d}_{zx} \\ \tau_{xy} = \mu \mathrm{d}_{xy} , \end{cases} \tag{10}$$

or in matrix form by

$$\{\sigma\} = [C]\{\mathrm{d}\} - \{p\} \tag{11}$$

in which

$$[C] = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ & & 2\mu+\lambda & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & \text{Symmetric} & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}, \quad \{p\} = \begin{Bmatrix} p \\ p \\ p \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{12}$$

In equations (10) (or (12)) μ is called the coefficient of viscosity, λ the bulk viscosity and p the static pressure.

For incompressible fluids the value of the bulk viscosity λ has no effect on the relationship between stresses and strain rates because those terms in equations (10) having λ as a multiplier disappear due to continuity equation (9). Consequently the constitutive law in incompressible flow is usually presented in the form

$$\begin{cases} \sigma_x = 2\mu d_x - p \\ \sigma_y = 2\mu d_y - p \\ \sigma_z = 2\mu d_z - p \\ \tau_{yz} = \mu d_{yz} \\ \tau_{zx} = \mu d_{zx} \\ \tau_{xy} = \mu d_{xy}, \end{cases} \tag{13}$$

which can be also interpreted as equation (10) with λ equal to zero. Here, however, the constitutive law (11) ($\lambda \neq 0$) is employed because the value of λ proves to have a considerable influence on the rate of convergence of the iterative solution method.

The solution for creeping flow has to satisfy equation (11), relation between strain rates and velocity components (1), equation of equilibrium (5) and equation of continuity (7). Substituting expression (1) for strain rates into equation (11) and then substituting the expression for stress components obtained into equations of equilibrium (5) we obtain it in terms of velocity components $\{\nu\}$ and pressure p as follows:

$$[\partial]^T[C][\partial]\{\nu\} - [\partial]^T\{p\} + \{F\} = \{0\}. \tag{14}$$

Equation (14) together with continuity equation

$$\{\nabla\}^T\{\nu\} = 0 \tag{7}$$

forms a system of four linear partial differential equations for the solution of the unknown functions ν_x, ν_y, ν_z and p.

The boundary conditions of the problem are given as known components of velocity or as known components of surface traction vector i.e.

$$\{v\} = \{\bar{v}\} \quad \text{on surface } S_v, \tag{15}$$

$$\{T\} = \{\bar{T}\} \quad \text{on surface } S_T. \tag{16}$$

Surface S_v is that portion of the boundary of the domain where the velocity components $\bar{v}_x$, $\bar{v}_y$ and $\bar{v}_z$ are known. Surface S_T is that portion where the traction components $\bar{T}_x$, $\bar{T}_y$ and $\bar{T}_z$ are known. The boundary conditions can naturally be given also in a mixed form for instance so that on some area the known quantities are for example $\bar{v}_x$, $\bar{T}_y$ and $\bar{T}_z$. Substituting expression (1) for the strain rates into equation (11) and the expression for the stress components obtained into equation (3), we have the components of the stress vector $\{T\}$ in equation (16) in terms of the basic unknowns $\{v\}$ and p. The boundary conditions for the system of partial differential equations (14) and (7) are now in their final form

$$\{v\} = \{\bar{v}\} \quad \text{on surface } S_v, \tag{15}$$

$$[n]^T[C]\,[\partial]\,\{v\} - [n]^T\{p\} = \{\bar{T}\} \quad \text{on surface } S_T. \tag{17}$$

3. *Solution of the equations of creeping flow by iteration*

The solution of the system of partial differential equations (14) and (7) subject to boundary conditions (15) and (17) is not possible in practical situations without use of discrete numerical methods. The discretization of the equations leads to a very large system of simultaneous linear algebraic equations, the solution of which demands much computer space.

Here an iterative solution scheme is introduced by which over half of the computer space needed in the solution of the system of the equations can be saved, thus making solution of problems having over double size possible within the same computer space. In this method the pressure $p = p(x, y, z)$ is determined iteratively so that the velocity components found from equations (14) finally satisfy the equation of continuity (9) with sufficient accuracy.

In incompressible flow a relation

$$p = -(\sigma_x + \sigma_y + \sigma_z)/3 \tag{18}$$

for the static pressure is obtained by employing equations (10) and the equation of continuity (9). By employing equation (18), a new value for the pressure p^{n+1} can be evaluated from the stress components σ_x^n, σ_y^n and σ_z^n calculated on the nth iteration. Substituting their expressions from equations (10), we obtain a new value for the pressure

$$p^{n+1} = p^n - \left(\frac{2}{3}\mu + \lambda\right) \mathrm{d}^n. \tag{19}$$

Writing d^n further in terms of velocity components $\{v\}^n$, we obtain the final equations used in the iteration process

$$[\partial]^T[C]\,[\partial]\,\{v\}^n - [\partial]^T\{p\}^n + \{F\} = \{0\} \tag{20}$$

$$\{v\}^n = \{\bar{v}\} \qquad \text{on surface } S_v, \tag{21}$$

$$[n]^T[C]\,[\partial]\,\{v\}^n - [n]^T\,\{p\}^n = \{\bar{T}\} \qquad \text{on surface } S_T, \tag{22}$$

$$p^{n+1} = p^n - \left(\frac{2}{3}\mu + \lambda\right) \{\nabla\}^T\{v\}^n, \tag{23}$$

$$\{p\}^{n+1} = p^{n+1}[1 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0]^T. \tag{24}$$

Taking a suitable starting value for p, for which we can take

$$p^1 = 0 \Rightarrow \{p\}^1 = \{0\}, \tag{25}$$

we can solve velocity components $\{v\}^n$ from equations (20), (21) and (22) for iterations $n = 1, 2, 3, \ldots$ and calculate after each iteration new values for p^{n+1} and $\{p\}^{n+1}$ from equations (23) and (24), until the results have converged sufficiently.

Some conclusions can be drawn on the effect of constant λ on the necessary number of iterations. From equations (10), there are found on the first iteration ($p^1 = 0$) the relationship

$$\sigma_x^1 + \sigma_y^1 + \sigma_z^1 = (2\mu + 3\lambda)\,\mathrm{d}^1, \tag{26}$$

from which there are found

$$\mathrm{d}^1 = \frac{\sigma_x^1 + \sigma_y^1 + \sigma_z^1}{2\mu + 3\lambda} \tag{27}$$

To obtain already on first iteration values of velocity components $\{v\}^1$, that would be as near as possible to the values of incompressible flow, the rate of dilatation d^1 calculated from $\{v\}^1$ should be as small as possible. On the basis of equation (27), the constant λ should be taken as high as possible. However, very high value for λ cannot be used, as numerical inaccuracies in the solution of equations (20), (21) and (22) appear. To find a safe upper limit for λ, we have taken recourse to the analogy between a Newtonian fluid and an elastic solid. Equations (20), (21) and (22) are completely analogous on the first iteration with the differential equations for the displacements of an isotropic linear elastic solid. In this analogy, to the coefficient of viscosity μ, there corresponds the shear modulus, and to the bulk viscosity λ, there corresponds the so called Lamé's constant. From numerical solutions of the differential equations for displacements of an isotropic linear elastic solid, there are obtained experience of the effect of Lamé's constant on the accuracy of the solutions. From these results a safe value for λ can be taken as

$$\lambda = 25\mu ... 100\mu \tag{28}$$

depending on the problem and on the accuracy of the computer.

4. *Discretization of the equations by the finite element method*

Equations (20) subject to boundary conditions (21) and (22) can be solved numerically using for example the finite difference or the finite element method. Because of its generality, the finite element method is used here.

The principles of the finite element method are explained thoroughly for example in [10]. Shortly the method is as follows. The domain under consideration is divided into sub-domains called elements, on the boundaries of which a certain number (total number = k) of so called nodal points or nodes are chosen. The velocity components at the nodes $v_{x1}, v_{y1}, v_{z1}, v_{x2},...,v_{xk}, v_{yk}, v_{zk}$ are the unknown quantities of the discretized problem compared with the unknown functions $v_x(x, y, z)$, $v_y(x, y, z)$, $v_z(x, y, z)$.

The displacement distribution is approximated in the form

$$\{v\} = [\psi]\{V\} \tag{29}$$

where

$$\{V\} = [v_{x1}, v_{y1}, v_{z1}, v_{x2},...,v_{xk}, v_{yk}, v_{zk}]^T. \tag{30}$$

The elements of the $[\psi]$ matrix are known functions of x, y and z, called shape functions. They are defined element by element and can obtain values differing from zero each one only in elements having certain node as a common point. On the basis of formula (29), there follows from equations (1) and (11)

$$\{d\} = [\partial][\psi]\{V\}, \tag{31}$$

$$\{\sigma\} = [C][\partial][\psi]\{V\} - \{p\}. \tag{32}$$

Formulae (31) and (32) determine the approximation for strain rates and stresses. It is obvious that stresses (32) cannot any more satisfy the equations of equilibrium (5) and boundary conditions (16) exactly with any value of vector $\{V\}$. However, equations (5) and (16) can be satisfied approximately for example by applying the Galerkin process [10]. When applying Galerkin's process to equation (5), a system of equations

$$\int_V [\psi]^T([\partial]^T\{\sigma\}+\{F\})\,\mathrm{d}V=\{0\}, \tag{33}$$

or

$$\int_V [\psi]^T[\partial]^T\{\sigma\}\,\mathrm{d}V+\int_V [\psi]^T\{F\}\mathrm{d}V=\{0\} \tag{34}$$

is formed which is satisfied, if $\{\sigma\}$ satisfies the equation of equilibrium (5) in the domain V under consideration. Integration by parts of the first term in (34) gives

$$\int_S [\psi]^T[\mathrm{n}]^T\{\sigma\}\mathrm{d}S-\int_V([\partial]\,[\psi])^T\{\sigma\}\mathrm{d}V+\int_V [\psi]^T\{F\}\mathrm{d}V=\{0\}, \tag{35}$$

and taking into account relation (3), we obtain

$$\int_V([\partial][\psi])^T\{\sigma\}\mathrm{d}V=\int_V [\psi]^T\{F\}\mathrm{d}V+\int_S [\psi]^T\{T\}\mathrm{d}S. \tag{36}$$

Substituting approximation (32) for $\{\sigma\}$ into equation (36) and taking into account that on surface S_T of S boundary condition (16) is valid, we obtain further

$$\int_V([\partial]\,[\psi])^T[C]([\partial]\,[\psi])\,\mathrm{d}V\cdot\{V\}$$
$$=\int_V [\psi]^T\{F\}\mathrm{d}V+\int_{S_T}[\psi]^T\{\overline{T}\}\,\mathrm{d}S+\int_{S_v}[\psi]^T\{T\}\mathrm{d}S+\int_V([\partial][\psi])^T\{p\}\,\mathrm{d}V. \tag{37}$$

The unknowns in the system of linear equations represented by matrix equation (37) are the velocity components $v_{x1}, v_{y1}, v_{z1}, v_{x2},\ldots,v_{xk}, v_{yk}, v_{zk}$ of vector $\{V\}$ and the non-zero components of vector $\int_{S_v}[\psi]^T\{T\}\mathrm{d}S$. However, boundary conditions (15) on surface S_v have not yet been taken into account. Additional equations are obtained by equating the nodal velocity components at nodes on surface S_v to the known values at corresponding points from the boundary conditions. Closer examination shows that the non-zero terms in vector $\int_{S_v}[\psi]^T\{T\}\,\mathrm{d}S$ appear exactly in those equations in (37) which can be replaced by identities concerning the given velocity components. Taking this into account, the system of equations (37) can be changed into form

$$[K]\{V\}=\{R\}+\Delta\{R\} \tag{38}$$

where notation

$$[K]=\int_V([\partial]\,[\psi])^T[C]([\partial]\,[\psi])\,\mathrm{d}V, \tag{39}$$

$$\{R\} = \int_V [\psi]^T \{F\} \mathrm{d}V + \int_{S_T} [\psi]^T \{\overline{T}\} \mathrm{d}S, \tag{40}$$

$$\Delta\{R\} = \int_V ([\partial][\psi])^T \{p\} \mathrm{d}V \tag{41}$$

is employed and where the terms $\int_{S_v} [\psi]^T \{T\} \mathrm{d}S$ have been dropped. In (38) it must now be understood that certain rows are to be replaced with the given velocity identities. The system of linear, algebraic equations (38) forms a substitute for the system of linear partial differential equations (20) subjected to boundary conditions (21) and (22), from which the vector of nodal velocities $\{V\}^n$ can be determined when pressure p^n is known.

From equation (23), there follows for the pressure p^{n+1} the expression

$$p^{n+1} = p^n - \left(\frac{2}{3}\mu + \lambda\right) \{\nabla\}^T [\psi] \{V\}^n. \tag{42}$$

The iteration process is now as follows:
Start $p^1 = 0 \Rightarrow \{p\}^1 = \{0\}$.

1. Compute $\Delta\{R\}^n = \int_V ([\partial][\psi])^T \{p\}^n \mathrm{d}V$
2. Solve $\{V\}^n$ from equation
   $[K]\{V\}^n = \{R\}^n$
   in which $\{R\}^n = \{R\} + \Delta\{R\}^n$
3. Compute
   $p^{n+1} = p^n - \left(\frac{2}{3}\mu + \lambda\right) \{\nabla\}^T [\psi]\{V\}^n$
   and form $\{p\}^{n+1} = p^{n+1} [1\ 1\ 1\ 0\ 0\ 0]^T$
   for the next iteration.

Steps 1 to 3 are gone through for n = 1, 2, 3,... until the results have converged sufficiently.

5. *Numerical results*

Some results obtained by the finite element method are given below. The element type used is the isoparametric, curved, eight node quadrilateral element which has proved to be very efficient [10]. The integrations needed in the formulae are done numerically by Gaussian integration. The calculations have been executed in UNIVAC

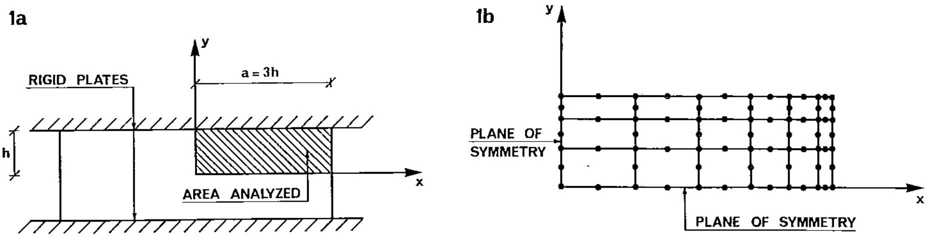


Figure 1. (a) Fluid squeezed between two rigid plates.
(b) Finite element mesh for the shaded area.

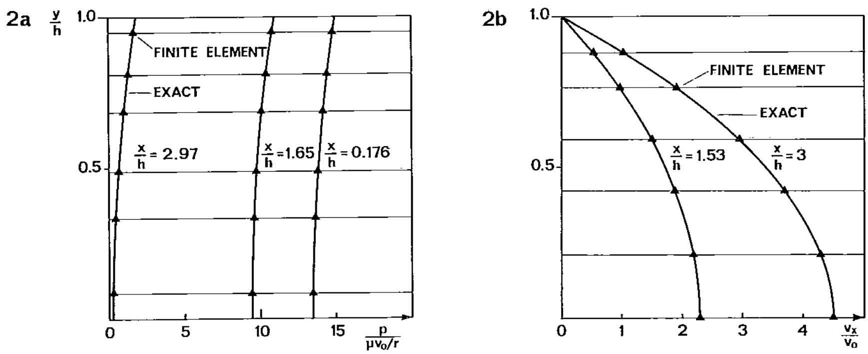


Figure 2. (a) Distribution of static pressure at three sections.
(b) Distribution of horizontal velocity at two sections.

1108 computer using a FORTRAN V program for plane and axi-symmetric cases developed by the writers.

The first example is concerned with a fluid squeezed between two rigid plates in plane flow case, figure 1a. The relative velocity between the plates is v_0. The results from the finite element analysis are compared with the known analytical solution [8]. The boundary conditions are naturally taken to be the same in both the solution methods. From symmetry reasons the area analyzed is the shaded region for which the finite element mesh (18 elements, 73 nodes) shown in figure 1b is used. The results for static pressure p at sections $x = 0 \cdot 176\,\text{h}$, $x = 1 \cdot 65\,\text{h}$ and $x = 2 \cdot 97\,\text{h}$, and the results for horizontal velocity v_x at sections $x = 1 \cdot 53\,\text{h}$ and $x = 3\,\text{h}$ are compared with the analytical solution. As seen from figure 2, the results are very accurate.

In connection with this example the effect of constant λ on the rate of convergence was also studied. This was done by comparing the change of velocities to the total velocities by employing as a measure the quantity

$$e^n = \frac{(\{V\}^n - \{V\}^{n-1})^T (\{V\}^n - \{V\}^{n-1})}{(\{V\}^n)^T \{V\}^n}, \quad n \geqslant 2. \tag{43}$$

Iteration was stopped when e^n was under 10^{-4} or when the number of iterations n was over 30. The number of iterations and the execution time are given in table 1 for some values of ratio λ/μ

Table 1. Effect of ratio λ/μ on the number of iterations and on the execution time.

λ/μ	No. of iterations	Execution time secs.	Comments
0	30	25.30	No convergence
1	30	25.76	No convergence
5	23	23.83	
10	16	17.76	
50	7	12.79	
100	6	12.47	
500	4	10.84	
1000	4	11.26	

It can be seen that the use of a value of λ differing from zero is necessary for convergence in this case. Thus, if we had used in the derivation of equations (20) to (24) the conventional constitutive law (13) instead of law (10), the method of solution had not succeeded in connection with this example.

The second example deals with the movements of the earth's crust and mantle under the loading ('the sea pressure' [2] to [5]) Δq resulting from erosion and sedimentation in assumed plane flow case. As shown in figure 3a, loading Δq was assumed to be distributed uniformly along the ocean and continent and to vary linearly at the coastal area. It is assumed that there are four separate horizontal layers the lowest one being rigid. The viscosities of the other layers from up to down are $\mu = 10^{22}$, $\mu = 10^{21}$ and $\mu = 10^{23}$ g/cms. The element mesh used is shown in figure 3b (70 elements, 245 nodes). The results for horizontal velocity v_x at some vertical sections is shown in figure 3c and the distribution of vertical velocity v_y at some horizontal sections is shown in figure 3d. KUTVONEN [4] has calculated a very similar example using a Fourier-series solution. The distribution of loading Δq was, however, due to the solution method a little wavy (see figure 3a), and the viscosity of the lowest layer had a finite value $\mu = 10^{25}$ g/cms. Results by the

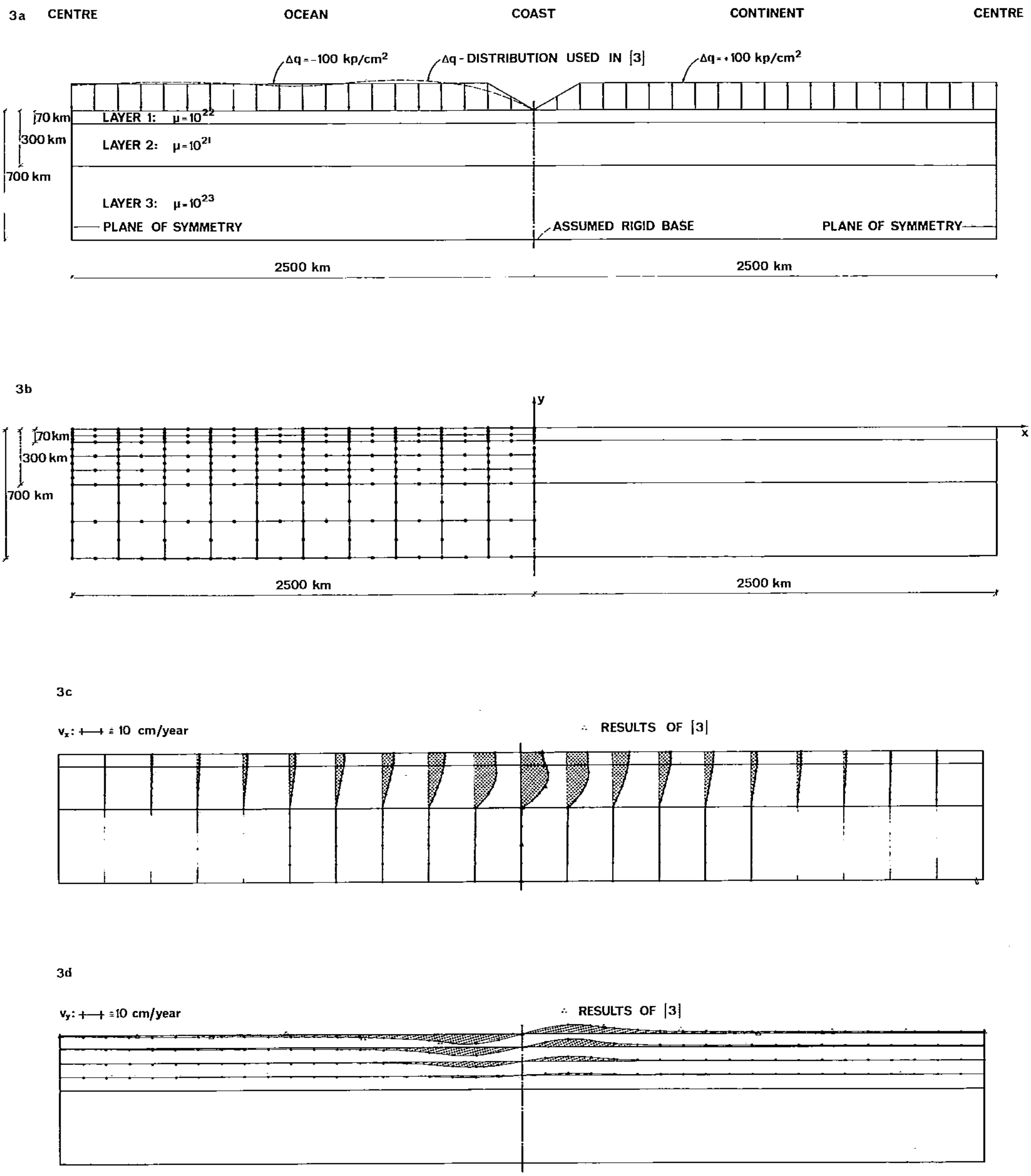


Figure 3. (a) Geometry and distribution of loading and viscosity.
(b) Finite element mesh used.
(c) Distribution of horizontal velocity.
(d) Distribution of vertical velocity.

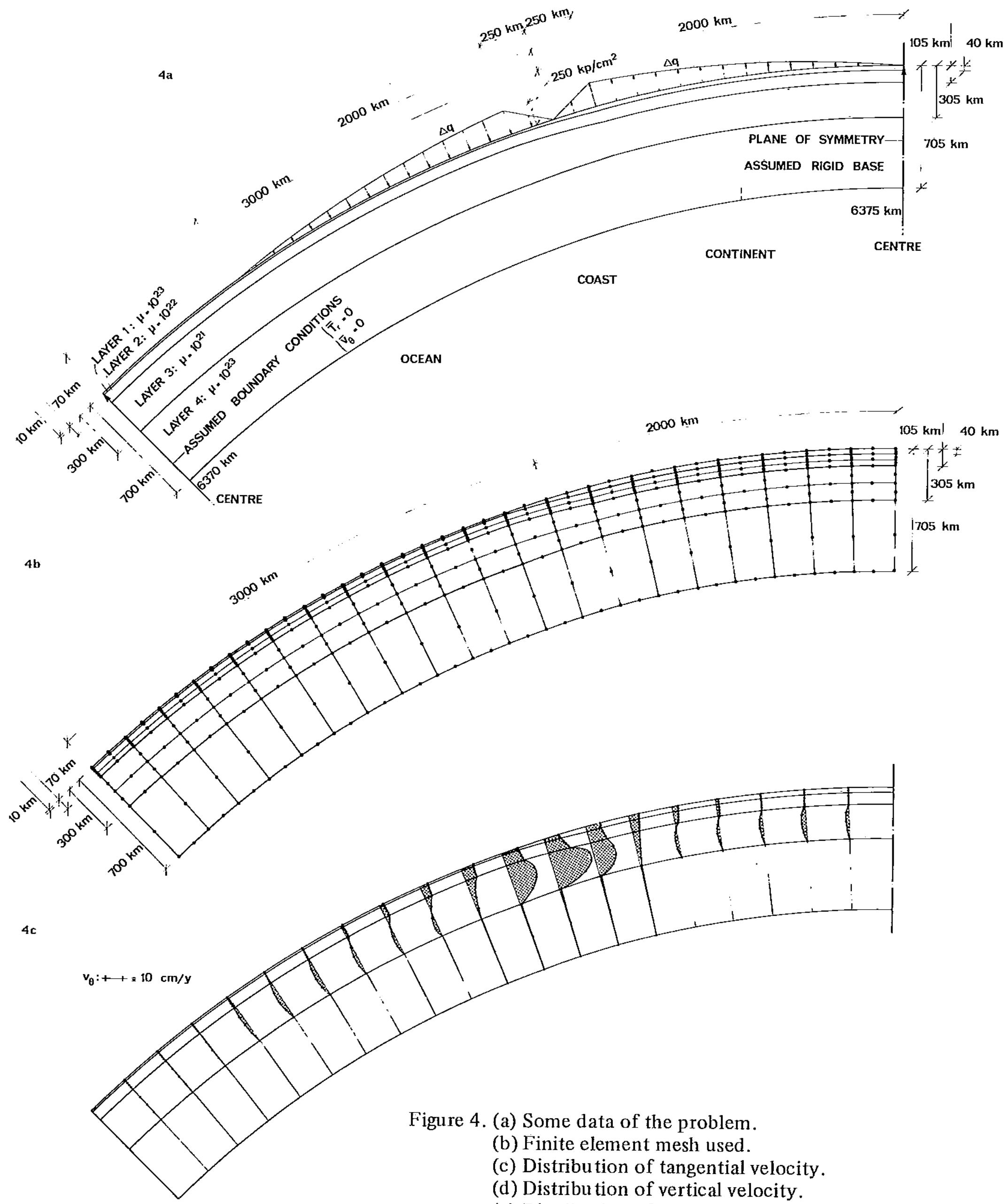
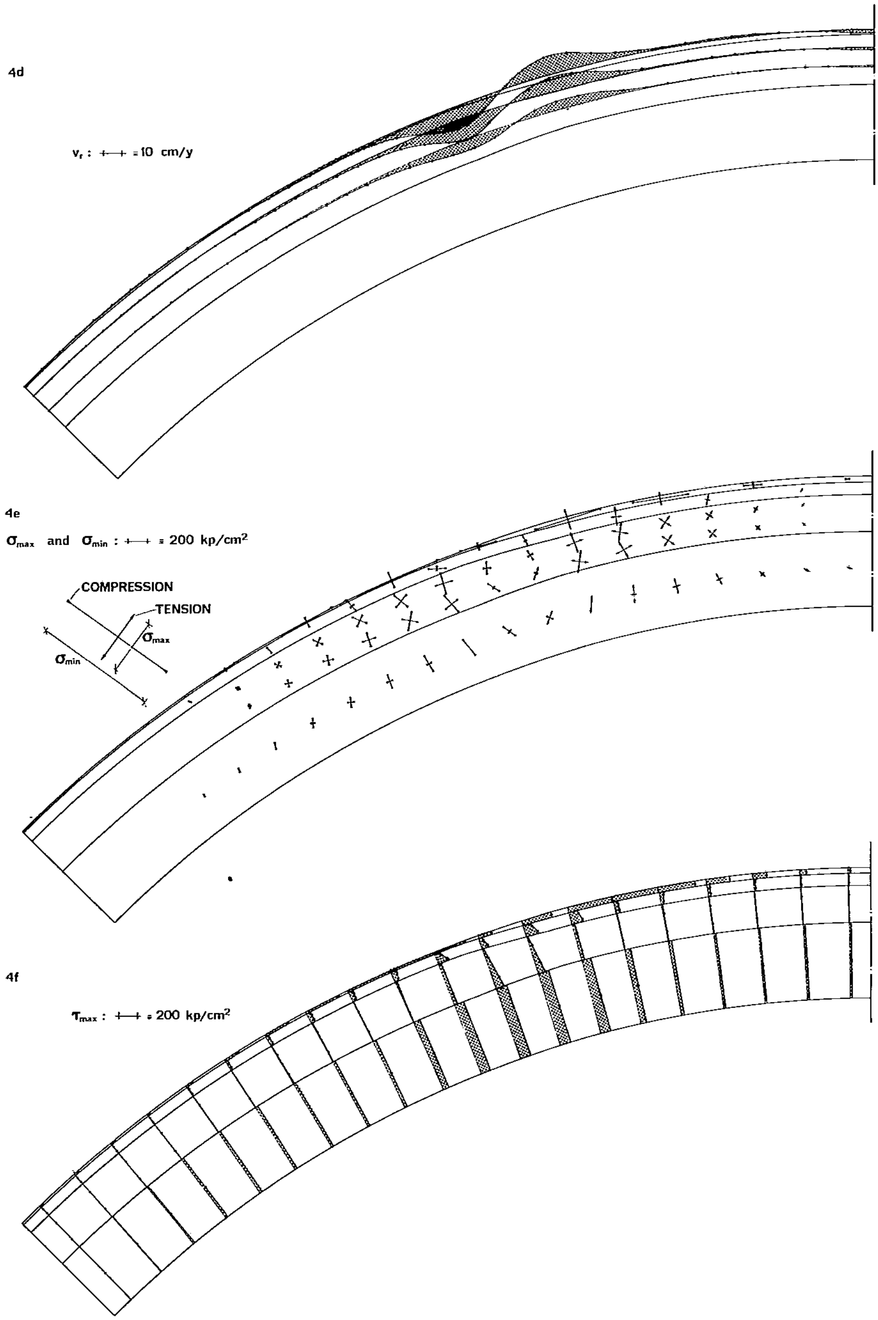


Figure 4. (a) Some data of the problem.
(b) Finite element mesh used.
(c) Distribution of tangential velocity.
(d) Distribution of vertical velocity.
(e) Distribution of principal stresses.
(f) Distribution of maximum shear stress.


4d
v_r : +—+ = 10 cm/y
4e
σ_{max} and σ_{min} : +—+ = 200 kp/cm²
COMPRESSION
TENSION
σ_{max}
σ_{min}
4f
τ_{max} : +—+ = 200 kp/cm²

finite element method have been compared with the results by the series solution [4] in figures 3c and 3d. Small differencies, resulting evidently partly from the differencies in load distributions, can be detected.

The third example is concerned with the determination of movements and stresses in the earth's crust and mantle taking into account the spherical shape of the earth, figure 4a. A more realistic load distribution due to erosion and sedimentation has also been tried to achieve in this example. Likewise, the stratification with respect to viscosity differs below the ocean and continent. The problem has been analysed as an axi-symmetric case with the axis of rotation horizontal in figure 4a. From figure 4a the rest of the important data of the problem can also be found. The element mesh (120 elements, 413 nodes) is shown in figure 4b. In figures 4c and 4d the distribution of tangential and radial velocities v_θ and v_r can be seen. The distribution of the principal stresses $\sigma_{\max}$ and $\sigma_{\min}$ and the maximum shear stress $\tau_{\max}$ are presented in figures 4e and 4f.

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Notations

$[C]$	stress-strain rate matrix, (11)
d	rate of dilatation
$d_x, d_y, d_z,$ d_{yz}, d_{zx}, d_{xy}	strain rates
$\{d\}$	column vector of strain rates, (2)
$\vec{F}$	body force vector per unit volume
F_x, F_y, F_z	body force components
$\{F\}$	column vector of body force components, (6)
k	total number of nodes
$[K]$	coefficient matrix in the system of equations (38), (39)
n	superscript referring to iteration number
$\vec{n}$	unit normal vector to a surface
n_x, n_y, n_z	components of unit normal vector
$[n]$	matrix of the components of the unit normal vector, (4)
p	static pressure
$\{R\}$	column vector in the system of equations (38) due to external forces, (40)

$\Delta\{R\}$	column vector in the system of equations (38) due to static pressure, (41)
S	surface of the domain under consideration
S_T	portion of the surface over which the traction components are known
S_v	portion of the surface over which the velocity components are known
T	superscript referring to transpose of a matrix
$\vec{T}$	stress vector
T_x, T_y, T_z	components of stress vector
$\{T\}$	column vector of the components of the stress vector, (4)
$\bar{T}_x, \bar{T}_y, \bar{T}_z$	given traction components
$\{\bar{T}\}$	column vector of traction components
$\vec{v}$	velocity vector
v_x, v_y, v_z	velocity components
$\{v\}$	column vector of velocity components, (2)
$\bar{v}_x, \bar{v}_y, \bar{v}_z$	known velocity components
$\{\bar{v}\}$	column vector of the known velocity components
V	domain under consideration
$\{V\}$	vector of nodal velocity components, (30)
x, y, z	rectangular Cartesian co-ordinates
$[\partial]$	matrix of partial differential operators, (2)
$\{\nabla\}$	column vector of partial differential operators, (8)
λ	bulk viscosity
μ	coefficient of viscosity
$\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy}$	stress components
$\{\sigma\}$	column vector of stress components, (4)
$[\psi]$	matrix of shape functions, (29)

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