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DETECTION OF SIGNAL POWER IN THE PRESENCE OF NOISE

by

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Abstract

The average power spectrum of N data sections containing one and the same weak signal superposed on different realisations of background noise is compared to the average power spectrum of M realisations of the pure noise. Assuming independence of the power spectrum estimates an exact F test is given for the presence of a signal at some given frequency. The power of the test is obtained from the noncentral F-distribution and its tabulated for selected M, N and signal-to-noise ratios r. Also, the values of r yielding the test power 0.9 for given M and N are tabulated. If the spectra are locally white, the exact test may be applied also to spectra smoothed over frequency. For non-white spectra smoothed over frequency an approximate F-test and its approximate power are given.

1. *Introduction*

Estimation of power spectrum is a commonly used method in analysis of geophysical observations. However, when the signal intended for analysis rises from a noise background, the contribution of the undesired noise to the computed spectrum should be estimated prior to drawing conclusions.

A seemingly good signal-to-noise ratio at the dominant frequency of the signal does not guarantee that signal power is present in the whole frequency range on which conclusions will be based.

In the following discussion it will be assumed that the noise is a stationary, random Gaussian process with zero mean, uncorrelated with the signal. Also, it is assumed that there exists records of noise with no signal present. This is the case of analysis of seismic signals, since they are transient, and since pure noise (which is approximately stationary within the time scale of earthquake signals) is usually recorded prior to the signal onset.

Computation of the noise power spectrum, in addition to the signal power spectrum, and comparison of the two spectra, gives a possibility to estimate the probability of the spectral power noticed in some frequency band being caused entirely by noise.

2. *Correction for noise power*

We write $Z(f)$ for the power spectrum of a record section which is supposed to contain a transient signal together with noise (f = frequency). The Fourier transform of the signal, which is taken to be deterministic, is $s(f)$, and its power spectrum is $|s(f)|^2$. We denote by $Y(f)$ the estimate of the power spectrum of the noise, obtainable from a record section preceding the signal. For noise with the properties defined above, the power estimate $Y(f)$ is distributed as $c_f \chi_2^2$ where c_f is an unknown constant [1].

An estimate of the signal power is obtained by the subtraction

$$P(f) = Z(f) - Y(f), \tag{1}$$

The expected value of $P(f)$ is according to LACOSS and KUSTER [4]

$$E(P) = |s|^2 \tag{2}$$

where, for brevity, the implicit dependence on frequency is not written out. Thus the estimate (1) of the signal power contains no bias due to noise. The stability of the estimate has been discussed by LACOSS and KUSTER [4], and can be computed if the signal-to-noise ratio $|s|^2/E(Y)$ is known. If we have only one observation for Z and Y, the stability is very low.

3. *Testing whether signal power is present*

The stability of the estimates of Z and Y can be improved by ensemble averaging and frequency smoothing [1]. Considering ensemble averaging, if there are available N parallel channels recording the same signal but different realizations of noise, as is the case with a seismic array station, the signal and noise section stabilities can be improved by averaging over the power spectra computed for different channels.

Because several noise sections can be analyzed from each channel, the number M of noise sections can be larger than N. The noise is here assumed to be an ergodic process. The averages are

$$Z = \frac{1}{N} \sum_{n=1}^{N} Z_n \qquad Y = \frac{1}{M} \sum_{m=1}^{M} Y_m \tag{3}$$

Table 1. Critical values for the ratio Z/Y in the test for presence of a signal. Upper values: $\alpha = 0.05$, lower values $\alpha = 0.01$.

M \ N	1	2	3	4	10	30	100
1	19.0 99.0	19.2 99.2	19.3 99.3	19.4 99.4	19.5 99.4	19.5 99.5	19.5 99.5
2	6.94 18.0	6.39 16.0	6.16 15.2	6.04 14.8	5.80 14.0	5.69 13.7	5.65 13.5
3	5.14 10.9	4.53 9.15	4.28 8.47	4.15 8.10	3.87 7.40	3.74 7.06	3.69 6.93
4	4.46 8.65	3.84 7.01	3.58 6.37	3.44 6.03	3.15 5.36	3.01 5.03	2.95 4.91
10	3.49 5.85	2.87 4.43	2.60 3.87	2.45 3.56	2.12 2.94	1.95 2.61	1.88 2.48
30	3.15 4.98	2.53 3.65	2.25 3.12	2.10 2.82	1.75 2.20	1.53 1.84	1.44 1.68
100	3.04 4.71	2.42 3.41	2.14 2.89	1.98 2.60	1.62 1.97	1.39 1.58	1.26 1.39

Because the Y_m's are independent random variables their sum MY is distributed as $c_f\chi^2_{2M}$. The Z_n's also are mutually independent, and independent of the Y_m's; in case $|s|^2 = 0$, NZ is distributed as $c_f\chi^2_{2N}$. In the ratio Z/Y the unknown c_f:s cancel out and the quotient is $F_{2N,2M}$ distributed. If $|s|^2 > 0$, Z and Z/Y tend to be larger than in the case $|s|^2 = 0$, and the test becomes:

Reject the null hypothesis $|s|^2 = 0$ if $Z/Y > F_{2N,2M}(\alpha)$

Table 1 gives the values of $F_{2N,2M}(\alpha)$ for some M,N and for $\alpha = 0.05, 0.01$.

Example: If $M = 10, N = 4$ and $\alpha = 0.05$ we reject $|s|^2 = 0$ if $Z/Y > 2.45$. If in fact $|s|^2 = 0$, we then erroneously reject it in 5 % of all cases. It may be noted that the normal approximation used in ref. [5] leads to an incorrect critical value of Z/Y.

If the noise spectrum is at least locally white, M and N can be increased by a factor K by averaging the power over K neighbouring frequencies as well as over the ensemble. In case of a non-white noise spectrum a similar increase in the number of degrees of freedom of the distribution of Y is not achieved.

Let the total average be

$$Y_A = \frac{1}{K} \sum_{k=1}^{K} Y(f_k)$$

where every $Y(f_k)$ is the ensemble average of M independent observations. Then for every k, $MY(f_k)$ is distributed as $c_k \chi^2_{2M}$ where c_k is an unknown constant depending on k. Hence MY_A is of the form

$$\sum_{k=1}^{K} c_k \chi^2_{2M}$$

which is not of $c\chi^2$-type unless all c_k are equal. However, a standard procedure in such cases [2] is to approximate the distribution by a $c\chi^2$ distribution with mean and variance equal to those of MY_A.

According to the properties of the χ^2 distribution, a suitable unbiased estimate of $D^2 Y(f_k)$ is

$$\frac{Y^2(f_k)}{M+1}$$

and this gives as estimate for the number of degrees of freedom

$$f_Y = 2 \cdot \text{stability} = 2 \cdot \frac{\left(\sum_{k=1}^{K} Y(f_k)\right)^2}{\sum_{k=1}^{K} Y^2(f_k)} (M+1).$$

For NZ_A (based on N observations) the corresponding estimate for the number of degrees of freedom is, under the null hypothesis,

$$f_Z = \frac{N}{M} f_Y$$

and it follows that under the null hypothesis the ratio Z_A/Y_A is distributed approximately as $F\left(\frac{N}{M} f_Y, f_Y\right)$. Again, the critical region leading to rejection of the null hypothesis is determined according to the F distribution.

4. *Power of the test*

By the power of a test we mean, as usual, the probability of rejecting the null hypothesis when it is false. The argument on which this probability depends is in our case the signal-to-noise ratio

$$r = \frac{|s|^2}{E(Y)} .$$

For the case $N = 1$, let $s + y$ represent the Fourier transform of a record section containing a signal together with noise. Let y be the Fourier transform of the noise in the section.

Let

$$s + y = s_R + y_R + i(s_I + y_I),$$

thus

$$Z = |s + y|^2 = (s_R + y_R)^2 + (s_I + y_I)^2 .$$

y_I and y_R are independent and $N(0, c)$ distributed ($c = E(Y)/2$). We can write

$$Z = c\left[\left(\frac{s_R}{\sqrt{c}} + \frac{y_R}{\sqrt{c}}\right)^2 + \left(\frac{s_I}{\sqrt{c}} + \frac{y_I}{\sqrt{c}}\right)^2\right]$$

which shows that Z is distributed as $c\chi'^2_{2,|s|^2/c}$, i.e. as a noncentral χ^2 with non-centrality parameter $|s|^2/c = 2r$. For general N, NZ has the distribution $c\chi'^2_{2N,2Nr}$ and then Z/Y has the noncentral F distribution $F'_{2N,2M,2Nr}$. This makes it possible to find the power of the test used here. In Table 2 values of its power are given for some N,M and fixed r. In Table 3 the values of r yielding the test power 0.9 are given for some N, M.

Example: If $M = 10$, $N = 4$ and $\alpha = 0.05$ as in the previous example, Table 2 shows that signals with signal-to-noise ratios 2,4 and 8 are detected by our test with probabilities .65, .97 and >.99, respectively. Suppose we use $\alpha = 0.01$ (we wrongly reject $|s|^2 = 0$ in only 1 % of all cases). If we want to detect a signal with $r = 1.5$ with a probability $\geqslant 0.9$, Table 3 shows that

for $N = 15$ we must have $M \geqslant 100$
for $N = 30$ we must have $M \geqslant$ about 25

For a fixed $M + N$ the highest power is seen to occur when N is slightly less than M.

For finding the approximate power of our test in the case of frequency smoothing over a non-white spectrum, suppose we know the expected noise power $EY(f_k) = e_k$

Table 2. Power of the test for r = 2, 4, 8. Upper values: $\alpha = 0.05$, lower values: $\alpha = 0.01$.

	$N=1$			$N=2$			$N=4$		
	r=2	4	8	r=2	4	8	r=2	4	8
$M=3$	.26	.48	.79	.32	.58	.88	.35	.66	.93
	.07	.19	.42	.10	.24	.53	.12	.29	.63
$M=10$	.33	.65	.93	.50	.83	.99	.65	.97	>.99
	.15	.36	.75	.24	.57	.93	.30	.78	>.99
$M=30$	.39	.70	.95	.57	.89	>.99	.77	.98	>.99
	.18	.45	.83	.32	.70	.98	.53	.97	>.99

based on ref. [6].

Table 3. Values of r yielding the power 0.9 for the test. Upper values: $\alpha = 0.05$, lower values: $\alpha = 0.01$.

M \ N	3	4	6	10	15	30	100
3	7.6	7.1	6.5	6.2	6.0	5.8	5.6
	15.6	14.6	13.6	12.9	12.3	11.8	11.3
4	6.1	5.6	5.1	4.6	4.4	4.1	4.0
	11.6	9.9	9.1	8.2	7.9	7.5	7.2
6	4.9	4.3	3.7	3.3	3.1	2.8	2.6
	7.6	6.9	6.1	5.4	5.0	4.7	4.3
10	4.0	3.4	2.9	2.5	2.2	1.9	1.7
	6.4	5.2	4.4	3.6	3.2	2.9	2.6
15	3.6	3.1	2.5	2.1	1.8	1.5	1.3
	5.1	4.3	3.5	2.9	2.5	2.1	1.8
30	3.2	2.7	2.2	1.7	1.4	1.1	.83
	4.4	3.7	2.9	2.3	1.9	1.4	1.2
100	3.0	2.5	1.9	1.3	1.1	.82	.51
	4.0	3.4	2.6	1.9	1.5	1.1	.68

based on ref. [3]

(say) and signal-to-noise ratio $r_k = s_k^2/e_k$ for all appearing frequencies f_k. Then, approximating

$$\sum_{k=1}^{K} MY(f_k) \quad \text{by} \quad c'\chi^2_{f'_y}$$

in the usual way, we get

$$f'_y = 2M \frac{\sum_{k=1}^{K} e_k^2}{\sum_{k=1}^{K} e_k}$$

and similarly $f'_z = \frac{N}{M} f'_y$.

Equating the expected values of $\sum_{k=1}^{K} NZ(f_k)$ and a tentative approximation to it of form $c'\chi'^2_{f'_z,\lambda}$ we get

$$\lambda = 2N \frac{\sum_{k=1}^{K} e_k r_k \sum_{k=1}^{K} e_k}{\sum_{k=1}^{K} e_k^2}$$

To the previous r now corresponds

$$r' = \frac{\lambda}{f_Z} = \frac{\sum_{k=1}^{K} e_k r_k}{\sum_{k=1}^{K} e_k},$$

i.e., a weighted average of signal-to-noise ratios r_k. To find the power of the test use f'_Z, f'_y and r' as f_Z, f_y and r before.

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