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OSCILLATIONS OF A THERMOVISCOELASTIC, SELFGRAVITATING, SPHERICALLY SYMMETRIC AND ROTATING EARTH MODEL

by

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Abstract

The purpose of this work has been first to derive the equations for the infinitesimal oscillations of a thermoviscoelastic, selfgravitating, spherically symmetric and rotating earth model on the basis of the axiomatic treatment of modern continuum mechanics. Secondly, the synthetic seismograms for the model in question have been obtained by systematic use of the fundamental matrices $Y(r)$ and $Z(r)$ together with a perturbation technique.

Introduction

For a historical background as well as for a foundation for the present work the reader is referred to the following research works from the field of free and forced oscillations and tidal deformations of certain earth models:

RAYLEIGH [30], LOVE [22] and [23], TAKEUCHI [37] and PEKERIS *et. al.* [28] have studied the free oscillations of an elastic, radially inhomogeneous, isotropic and selfgravitating sphere (with hydrostatic initial state of stress). For modifications of this model to take into account transversely isotropic symmetry see BACKUS [5]. Forced oscillations of this model have been studied by ALTERMAN *et. al.* [2] and by SAITO [33]. As shown by BACKUS and GILBERT [3], PEKERIS *et. al.* [29], MACDONALD and NESS [24], GILBERT and BACKUS [12], and DAHLEN [9], rotation causes splitting of the spectrum. SAASTAMOINEN [31] and [32] has presented two further modifications to take account of viscoelastic and thermal effects. For tidal oscillations of the model see for example TAKEUCHI [37] and TAKEUCHI and SAITO [38].

The purpose of the present work is to give a systematic treatment of the free and forced oscillations of a thermoviscoelastic (rate type), isotropic or transversely isotropic, spherically symmetric and rotating earth model in addition to handling in some detail the tidal deformations of this model. The more detailed structure of the different chapters may be outlined as follows:

The chapter on *Notations* presents first some principles from the analysis of scalar, vector and tensor valued fields (represented in abstract and in coordinate forms respectively). The second part of this chapter reviews some concepts which are somehow connected with the deformation.

In the chapter on *Basic principles of thermomechanics* the principles of thermodynamics have been used to find to first order infinitesimals the Laplace transformed equations of motion of a thermoviscoelastic, self-gravitating, inhomogeneous and rotating body. The dynamic field quantities are allowed to suffer jump discontinuities on a finite number of internal surfaces and the initial state of stress has been assumed to be hydrostatic, although according to the recent satellite observations the equilibrium state of the stress deviates slightly from the hydrostatic one.

In the chapter dealing with *Equations of motion in a rotating coordinate system* a scalar decomposition of the different field quantities, combined with the expansion into series with respect to the spherical surface harmonics, leads to two coupled systems of ordinary differential equations, which with certain boundary conditions determine the coefficients with respect to the spherical surface harmonics (for spheroidal and toroidal oscillations as well as for tidal deformations).

In the chapter entitled *Solution of the equations of motion* the adjoint boundary value problem, together with a technique of contour integration, has been used to produce a residue representation for the solutions (*i.e.* to determine the coefficients with respect to the spherical surface harmonics). The rotation has been included to first order perturbation with respect to the parameter $\varepsilon = \frac{\omega}{p^0}$, where ω is the angular velocity of the rotation and p^0 is an eigenvalue of the unperturbed problem. After solving the coefficients with respect to the surface harmonics, the synthetic seismograms are readily obtained. At the end of this chapter two numerical methods for the determination of the fundamental matrices have been studied in some detail.

At the end of the work three appendices have been added to support the main part of the study. *Appendix A* handles in a more detailed way the constitutive theory giving to first order infinitesimals the Laplace transformed constitutive relations between different field quantities. In *Appendix B* the Laplace transformed equation of energy has been derived to first order infinitesimals. Finally *Appendix C* presents some facts about formal solution of boundary value problems, which have been expressed by a system of ordinary differential equations with certain boundary conditions (both in matrix form).

Notations

In this chapter a few short notions about scalar, vector and tensor valued fields will be presented together with certain concepts about deformation.

Scalars: Scalars are identified with the elements of the space of real numbers $\Re$.

Scalar fields are regarded as mappings form the Euclidean space E (or some other space, *e.g.* H in (15)) into $\Re$. In other words

$$\psi = \hat{\psi}(\mathrm{x}) \; ; \mathrm{x} \in E \, , \quad \psi \in \Re \, .$$

The norm of an arbitrary element $\psi \in \Re$ is defined as the absolute value of ψ, or

$$|\psi|_{\Re} = |\psi| \, . \tag{1}$$

Vectors: Following NOLL [26] vectors are defined as the translation space V of the ordinary three dimensional Euclidean space E. Thus two elements x and y define a vector $\mathrm{v} \in V$ by

$$\mathrm{v} = \mathrm{y} - \mathrm{x} \, . \tag{2}$$

According to (2) vectors may be regarded as mappings from E into V. I.e. there is a transformation $\hat{\mathrm{v}}$ from E into V, such that

$$\mathbf{v} = \hat{\mathbf{v}}(\mathbf{x}) .$$

More general fields are obtained if instead of E the domain of $\hat{\mathbf{v}}$ is in another function space (*e.g.* H in (15)).

The scalar product in V is defined as a certain bilinear transformation $[\cdot , \cdot]$ from the product space $V \times V$ into $\Re$. (For product spaces and many other questions on modern analysis the reader is referred to the book by DIEUDONNÉ [10]). Thus, given two arbitrary elements $\mathbf{u}$ and $\mathbf{v}$, the scalar product between them is

$$[\mathbf{u} , \mathbf{v}] = \mathbf{u} \cdot \mathbf{v} . \tag{3}$$

The norm of an element $\mathbf{v} \in V$ is

$$|\mathbf{v}|_V = [\mathbf{v} , \mathbf{v}]^{1/2} = (\mathbf{v} \cdot \mathbf{v})^{1/2} \tag{4}$$

In E the distance between two elements $\mathbf{x}$ and $\mathbf{y}$ is defined by the Euclidean norm

$$|\mathbf{x} - \mathbf{y}|_E = ((\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}))^{1/2} . \tag{5}$$

Tensors: Tensors are regarded as equivalent to the elements of the space of linear transformations $L(V ; V)$ from V into V. Thus $\mathbf{v} \in V$ and $\mathbf{T} \in L(V ; V)$ define an element $\mathbf{u}$ by

$$\mathbf{u} = \mathbf{T}[\mathbf{v}] .$$

Tensor fields are defined as mappings from other function spaces into $L(V ; V)$.

The scalar product between two elements $\mathbf{A}$ and $\mathbf{B} \in L(V ; V)$ is given by the bilinear transformation

$$[\mathbf{A} , \mathbf{B}] = tr(\mathbf{A}\mathbf{B}^T) ; \ \forall \ \mathbf{A} , \mathbf{B} \in L(V ; V) \tag{6}$$

from $L(V ; V) \times L(V ; V)$ into $\Re$. In (6) tr means the trace of the quantity inside the brackets and $\mathbf{B}^T$ means the transpose of $\mathbf{B}$.

Higher order tensors are defined as spaces of certain linear transformations. For example, third order tensors are elements of $L(L(V ; V) ; V)$ (*i.e.* linear transformations from $L(V ; V)$ into V).

Coordinate representations of vectors: For many questions explicit coordinate representations are necessary (e.g. to obtain scalar decompositions of different field quantities).

It is easy to see that the three vectors defined by

$$\mathbf{e}_i = \partial_i \, \mathbf{x} ; \ \mathbf{e}_i \in V \tag{7}$$

form a base in V. In (7) ∂_i means the partial derivative with respect to the coordinate x^i.

The dual base $\{\mathbf{e}^j\}$ of $\{\mathbf{e}_i\}$ is obtained from the relations

$$\mathbf{e}^j \cdot \mathbf{e}_i = \delta_i^j , \tag{8}$$

where δ_i^j are the mixed components of the metric tensor $\mathbf{I} \in L(V\,;V)$.

The base systems $\{\mathbf{e}_i\}$ and $\{\mathbf{e}^j\}$ give for a vector $\mathbf{v} \in V$ the two coordinate representations

$$\mathbf{v} = v^i \mathbf{e}_i = v_j \mathbf{e}^j . \tag{9}$$

With the aid of (8) the coordinates v^i and v_j in (9) are seen to be related by the expressions

$$v_i = g_{ij} v^j \quad \text{and} \quad v^i = g^{ij} v_j , \tag{10}$$

where the covariant and contravariant components of the metric tensor $\mathbf{I}$ are respectively

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \quad \text{and} \quad g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j . \tag{11}$$

Coordinate representations of tensors: It is seen that each of the tensor products

$$\{\mathbf{e}^i \otimes \mathbf{e}^j\}, \{\mathbf{e}^i \otimes \mathbf{e}_j\}, \{\mathbf{e}_i \otimes \mathbf{e}^j\}, \quad \text{and} \quad \{\mathbf{e}_i \otimes \mathbf{e}_j\} \tag{12}$$

span $L(V\,;V)$. Consequently an arbitrary tensor $\mathbf{T} \in L(V\,;V)$ has the four different representations

$$\mathbf{T} = T_{ij}\, \mathbf{e}^i \otimes \mathbf{e}^j = T_i^j \mathbf{e}^i \otimes \mathbf{e}_j = T_j^i \mathbf{e}_i \otimes \mathbf{e}^j = T^{ij} \mathbf{e}_i \otimes \mathbf{e}_j . \tag{13}$$

From (13) relations similar to (10) may be obtained between the different components of $\mathbf{T}$.

Coordinate forms of the metric tensor $\mathbf{I}$ are given by

$$\mathbf{I} = g_{ij} \mathbf{e}^i \otimes \mathbf{e}^j = \delta_i^j \mathbf{e}^i \otimes \mathbf{e}_j = g^{ij} \mathbf{e}_i \otimes \mathbf{e}_j , \tag{14}$$

where the components have been given by (8) and (11).
[Remark: As is usual in the theory of tensors, repeated indices, one upper and one lower, imply summation over that index: from one to three in the case of Roman letters and from one to two in the case of Greek letters.]

Calculus of scalar, vector and tensor fields: Later we shall require some facts about the calculus of scalar, vector and tensor valued fields defined on product spaces of the form,

$$H = L(V_t\,;\,V_t) \times \Re_t \times V_t\,. \tag{15}$$

In (15) the letter t shows that the elements of the respective spaces depend upon the real parameter t (time). In what follows the fields ψ, $\mathbf{q}$ and $\mathbf{T}$ (all defined on H and having their ranges in $\Re_t$, V_t and $L(V_t\,;\,V_t)$ respectively) may serve as examples of the class of fields in question. Accordingly

$$\begin{aligned} \psi &= \hat{\psi}(\mathbf{c}\,,\vartheta\,,\mathbf{h}) \\ \mathbf{q} &= \hat{\mathbf{q}}(\mathbf{c}\,,\vartheta\,,\mathbf{h}) \\ \hat{\mathbf{T}} &= \hat{\mathbf{T}}(\mathbf{c}\,,\vartheta\,,\mathbf{h})\,. \end{aligned} \tag{16}$$

Since only the time derivative of ψ and the Taylor series (to the first order in H) of ψ, $\mathbf{q}$ and $\mathbf{T}$ around the element

$$\underset{0}{H} = (\mathbf{I}\,,\vartheta_0\,,0) \in H \tag{17}$$

are needed in what follows, the treatment here has been restricted to those concepts.

The time derivative of ψ takes on the form,

$$\dot{\psi} = \partial_{\mathbf{c}}\hat{\psi}[\dot{\mathbf{c}}] + \partial_\vartheta\hat{\psi}\dot{\vartheta} + \partial_{\mathbf{h}}\hat{\psi}[\dot{\mathbf{h}}]\,, \tag{18}$$

where the dot means the time derivative, $\partial_{\mathbf{c}}\hat{\psi}[\,\cdot\,] \in L(L(V_t\,;V_t)\,;\Re_t)$, $\partial_\vartheta\hat{\psi} \in L(\Re_t\,;\,\Re_t)$ and $\partial_{\mathbf{h}}\psi[\,\cdot\,] \in L(V_t\,;\,\Re_t)$.

The Taylor series in question may be shown to have the forms

$$\begin{aligned} \psi &= \underset{0}{\hat{\psi}} + \underset{0}{\partial_{\mathbf{c}}\hat{\psi}}[\mathbf{c}-\mathbf{I}] + \underset{0}{\partial_\vartheta\hat{\psi}}(\vartheta-\vartheta_0) + \underset{0}{\partial_{\mathbf{h}}\hat{\psi}}[\mathbf{h}] + \langle\,\rangle \\ \mathbf{q} &= \underset{0}{\hat{\mathbf{q}}} + \underset{0}{\partial_{\mathbf{c}}\hat{\mathbf{q}}}[\mathbf{c}-\mathbf{I}] + \underset{0}{\partial_\vartheta\hat{\mathbf{q}}}(\vartheta-\vartheta_0) + \underset{0}{\partial_{\mathbf{h}}\hat{\mathbf{q}}}[\mathbf{h}] + \langle\,\rangle \\ \mathbf{T} &= \underset{0}{\hat{\mathbf{T}}} + \underset{0}{\partial_{\mathbf{c}}\hat{\mathbf{T}}}[\mathbf{c}-\mathbf{I}] + \underset{0}{\partial_\vartheta\hat{\mathbf{T}}}(\vartheta-\vartheta_0) + \underset{0}{\partial_{\mathbf{h}}\hat{\mathbf{T}}}[\mathbf{h}] + \langle\,\rangle\,. \end{aligned} \tag{19}$$

In (19) the »subscript» zero refers to the element $\underset{0}{H}$ (defined in (17)). In the linear transformations in (19):

$\underset{0}{\partial_{\mathbf{c}}\hat{\psi}}[\,\cdot\,] \in L(L(V_t\,;\,V_t)\,;\,\Re_t)$, $\underset{0}{\partial_\vartheta\hat{\psi}} \in L(\Re_t\,;\,\Re_t)$, $\underset{0}{\partial_{\mathbf{h}}\hat{\psi}}[\,\cdot\,] \in L(V_t\,;\,\Re_t)$

$\underset{0}{\partial_{\mathbf{c}}\hat{\mathbf{q}}}[\,\cdot\,] \in L(L(V_t\,;\,V_t)\,;\,V_t)$, $\underset{0}{\partial_\vartheta\hat{\mathbf{q}}} \in L(\Re_t\,;\,V_t)$, $\underset{0}{\partial_{\mathbf{h}}\hat{\mathbf{q}}}[\,\cdot\,] \in L(V_t\,;\,V_t)$

$\underset{0}{\partial_{\mathbf{c}}\hat{\mathbf{T}}}[\,\cdot\,] \in L(L(V_t\,;\,V_t)\,;L(V_t\,;\,V_t))$, $\underset{0}{\partial_\vartheta\hat{\mathbf{T}}} \in L(\Re_t\,;\,L(V_t\,;\,V_t))$ and

$\partial_{\mathbf{h}} \underset{0}{\hat{\mathbf{T}}}[\,\cdot\,] \in L(V_t\,;\,L(V_t\,;\,V_t))\,.$

The symbol $\langle\,\rangle$ in (19) means the higher than first order terms in H.

Because the explicit expressions for the functionals $\partial_{\mathbf{c}}\hat{\psi}\,[\mathbf{c}]$ and $\partial_{\mathbf{h}}\hat{\psi}[\mathbf{h}]$ are needed later, it is relevant to point out that the representation theorems for linear functionals in $L(V_t\,;\,V_t)$ and in V_t reveal that

$$\partial_{\mathbf{c}}\hat{\psi}[\mathbf{c}] = \mathrm{tr}\,(\partial_{\mathbf{c}}\hat{\psi}\mathbf{c}^T) \tag{20}$$

and

$$\partial_{\mathbf{h}}\hat{\psi}[\mathbf{h}] = \partial_{\mathbf{h}}\hat{\psi}\cdot\mathbf{h}\,, \tag{21}$$

respectively.

On space derivation (gradient) of coordinate representations: From (7) it is evident that the base vectors $\{\mathbf{e}_i\}$ are field quantities in general (mappings from E into V). Therefore, under space derivation of the coordinate representations (i.e. application of the operator

$$\partial_{\mathbf{x}} = \mathbf{e}^j\partial_j \tag{22}$$

to the coordinate forms of vector and tensor fields), it is necessary to have expressions for the partial derivatives of the base vectors with respect to the coordinates x^j.

Since $\{\partial_j\mathbf{e}_i\} \in V$, there is a system of real numbers $\{\Gamma^k_{ji}\} \in \Re$ (the Christoffel symbols) such that

$$\partial_j\mathbf{e}_i = \Gamma^k_{ji}\mathbf{e}_k\,. \tag{23}$$

As to the partial derivatives of the dual set $\{\mathbf{e}^i\}$, it is soon verified using (8) and (23) that

$$\partial_j\mathbf{e}^i = -\,\Gamma^i_{jk}\mathbf{e}^k\,. \tag{24}$$

After these preliminaries the application of (22) on vector and tensor fields, taking into account (23) and (24), gives the required space derivatives. For example, the gradient of the vector field $\mathbf{v} \in V$ is of the form

$$\partial_{\mathbf{x}}\mathbf{v} = v^i_{|j}\mathbf{e}_i \otimes \mathbf{e}^j = v_{i|j}\mathbf{e}^i \otimes \mathbf{e}^j\,. \tag{25}$$

For many details about space derivation see SEDOW [36].

On the spherical coordinate system: Because of the spherical symmetry of the problem, it is necessary to present some characteristics of the differential geometry in this coordinate system. For a more complete treatment see the excellent work by BACKUS [5].

The position vector $\mathbf{x}$ is given by

$$\mathbf{x} = r\mathbf{x}^0 , \tag{26}$$

where r is the radius vector and $\mathbf{x}^0$ is the unit vector,

$$\mathbf{x}^0 \cdot \mathbf{x}^0 = 1 . \tag{27}$$

For a complete base system two other linearly independent vectors (besides $\mathbf{x}^0$) are needed. For this purpose a surface coordinate set $\{u^\alpha\}$ will be fixed (consequently the triplet (r, u^1, u^2) determines every point of E). Accordingly (as in (7)), the additional basis $\{\mathbf{a}_\alpha\}$, which spans the unit sphere S_1, is given by

$$\mathbf{a}_\alpha = \partial_\alpha \mathbf{x}^0 , \tag{28}$$

where ∂_α means the partial derivative with respect to the coordinate u^α.

The dual basis $\{\mathbf{a}^\beta\}$ of $\{\mathbf{a}_\alpha\}$ is obtained from

$$\mathbf{a}^\beta \cdot \mathbf{a}_\alpha = \delta^\beta_\alpha , \tag{29}$$

where δ^β_α are the mixed components of the surface metric tensor.

The operator (22) in the spherical coordinate system takes the form

$$\partial_\mathbf{x} = \mathbf{x}^0 \partial_r + \frac{\mathbf{a}^\gamma}{r} \partial_\gamma . \tag{30}$$

In principally the same way as above ((23) and (24)) the partial derivatives $\{\partial_\beta \mathbf{a}_\alpha\}$ and $\{\partial_\beta \mathbf{a}^\alpha\}$ are shown to satisfy the relations

$$\partial_\beta \mathbf{a}_\alpha = \Gamma^\gamma_{\beta\alpha} \mathbf{a}_\gamma - a_{\beta\alpha} \mathbf{x}^0 \tag{31}$$

and

$$\partial_\beta \mathbf{a}^\alpha = - \Gamma^\alpha_{\beta\gamma} \mathbf{a}^\gamma - \delta^\alpha_\beta \mathbf{x}^0 , \tag{32}$$

where $\{a_{\alpha\beta}\}$ are the covariant components of the surface metric tensor and $\{\Gamma^\gamma_{\beta\alpha}\}$ are the surface Christoffel symbols.

Some concepts on finite deformation: Under this title we consider some kinematical quantities which one way or another are connected with the time dependent transformation

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) , \tag{33}$$

called deformation (motion). Equation (33) is understood as a mapping from the Euclidean space E (material, Lagrangean) into the time dependent Euclidean space E_t (spatial, Eulerian).

Based on E and E_t two vector and tensor spaces are constructed. These are called V and $L(V; V)$ respectively in the case of the material system E, and V_t and $L(V_t; V_t)$ respectively in the case of the spatial system E_t.

The velocity $\mathbf{v}$ in V_t is equivalent to the material time derivative of $\mathbf{x}$, or in other words

$$\mathbf{v} = \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} \in V_t . \tag{34}$$

The acceleration $\mathbf{a}$ is the material time derivative of $\mathbf{v}$. Thus

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{L}\mathbf{v} \in V_t , \tag{35}$$

where $\mathbf{L}$ is the velocity gradient given by

$$\mathbf{L} = \partial_{\mathbf{x}} \mathbf{v} \in L(V_t; V_t) . \tag{36}$$

The deformation gradient $\mathbf{F}$ is defined by

$$\mathbf{F} = \partial_{\mathbf{X}} \mathbf{x} . \tag{37}$$

With the aid of the polar decomposition of (37) (see for example TRUESDELL and NOLL [40])

$$\mathbf{F} = \mathbf{R}\mathbf{U} , \tag{38}$$

where $\mathbf{R}$ is orthogonal ($\mathbf{R}\mathbf{R}^T = \mathbf{I}$) and $\mathbf{U}$ is symmetric ($\mathbf{U} = \mathbf{U}^T$).

The right Cauchy-Green tensor $\mathbf{C}$ is given by

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 . \tag{39}$$

The relation

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{40}$$

between the velocity gradient and the deformation gradient may easily be verified by taking the material time derivative of (37) and using (36).

Infinitesimal theory: Since the main concern of the present paper is with the infinitesimal relations, the deformation (33) will be replaced by

$$\mathbf{x} = \mathbf{X} + \mathbf{u} . \tag{41}$$

The deformation gradient, by (37) and (41), now takes the form

$$\mathbf{F} = \mathbf{I} + \mathbf{H}\,, \tag{42}$$

where the displacement gradient

$$\mathbf{H} = \partial_\mathbf{X} \mathbf{u}\,. \tag{43}$$

The infinitesimal form of the velocity gradient is obtained from (40) with the aid of (42) as follows:

$$\mathbf{L} = \dot{\mathbf{H}} + \langle\,\rangle\,. \tag{44}$$

As to $\mathbf{C}$, it follows from (39) and (42) that

$$\mathbf{C} = \mathbf{I} + 2\mathbf{E} + \langle\,\rangle\,, \tag{45}$$

where $\mathbf{E}$ is the material strain tensor given by

$$\mathbf{E} = \tfrac{1}{2}\,(\mathbf{H} + \mathbf{H}^T)\,. \tag{46}$$

Spatial representations: Many times it is necessary to change the domain of the field quantities from E into E_t (i.e. from material into spatial systems). As a result of such changes certain material field quantities will be given here in their spatial forms (to first order infinitesimal quantities in the respective spaces).

Acceleration (35) takes the form (47) after use of (41) and (44):

$$\mathbf{a} = \partial_t^2 \mathbf{u} + \langle\,\rangle\,. \tag{47}$$

The spatial and material displacement gradients ($\mathbf{h}$ and $\mathbf{H}$) are related by

$$\hat{\mathbf{h}} = \mathbf{H}\,\mathbf{F}^{-1}\,, \tag{48}$$

from which, using (42), it follows that

$$\hat{\mathbf{h}} = \mathbf{H} + \langle\,\rangle\,. \tag{49}$$

By (46) and (48) the spatial strain tensor $\mathbf{e}$ takes the form

$$\mathbf{e} = \mathbf{E} + \langle\,\rangle\,. \tag{50}$$

At the end of the discussion on deformation the domains of the fields $\varrho(\mathbf{X})$, $\mathbf{q}(\mathbf{X})$ and $\mathbf{T}(\mathbf{X})$ will be changed from E into E_t (from material into spatial representation). Thus $\mathbf{X}$ will be solved from (41), after which the subsequent Taylor series expansions around $\mathbf{x}$ result in

$$\varrho(\mathbf{X}) = \varrho(\mathbf{x}) - \mathbf{u}\cdot\partial_\mathbf{x}\varrho(\mathbf{x}) + \langle\,\rangle \tag{51}$$

$$\mathbf{q}(\mathbf{X}) = \mathbf{q}(\mathbf{x}) - \partial_{\mathbf{x}}\mathbf{q}[\mathbf{u}] + \langle\,\rangle \tag{52}$$

$$\mathbf{T}(\mathbf{X}) = \mathbf{T}(\mathbf{x}) - \partial_{\mathbf{x}}\mathbf{T}[\mathbf{u}] + \langle\,\rangle\,. \tag{53}$$

Laplace transformation: The transformation pair

$$(\;)\,(p) = \int_0^\infty e^{-pt}\,(\;)\,(t)dt \tag{54}$$

$$(\;)\,(t) = \frac{1}{2\pi i}\int_{LC} e^{pt}\,(\;)\,(p)dp \tag{55}$$

will be used in this paper to replace the time dependent field quantities by the parameter (p) dependent ones. In (55) $\int_{LC}$ means the usual Laplace contour integral (about the Laplace transformation see *e.g.* BÅTH [6]). In (54) and (55) the symbol ( ) may stand for any scalar, vector or tensor quantity.

Basic principles of thermomechanics

The derivation of the equations of motion of the thermoviscoelastic system will be based on the following fundamental principles, which express certain conservation and balance conditions between the different field quantities.

According to ERINGEN [11] these principles are:

— Conservation of mass
— Balance of momentum
— Balance of moment of momentum
— Balance of energy
— Clausius-Duhem inequality
— Constitutive axioms

A more detailed treatment of these principles will be given in the following sections.

Conservation of mass: Since the deformation (33) carries the body v_0 (see Fig. 1) into its deformed position v, the mass conservation condition expresses the physically obvious fact that the total mass of the body should be invariant under (33). In mathematical form the principle is

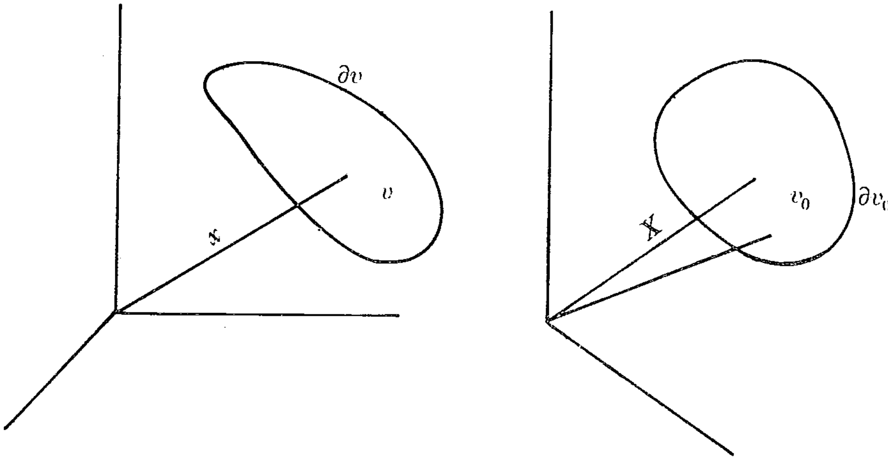

Fig 1.

$$\int_v \varrho dv = \int_{v_0} \varrho_0 dv_0 \,, \tag{56}$$

where $\varrho(\mathbf{x})$ and $\varrho_0(\mathbf{X})$ are the mass densities of the body in E_t and in E respectively. In this paper (56) is used in two local forms, which will be developed below.

The first local form is obtained by differentiating both sides of (56) materially with respect to time. It should be noted that the right-hand side of (56) is independent of t. Thus

$$\frac{d}{dt} \int_v dm = 0 \,, \tag{57}$$

where

$$dm = \varrho(\mathbf{x}) dv \,. \tag{58}$$

As to the second local form, it is found that the volume element dv_0 transforms under (33) as

$$dv = \det \mathbf{F} dv_0 \,. \tag{59}$$

Consequently, with the aid of (59), (56) is transformed into

$$\varrho_0(\mathbf{X}) = \det \mathbf{F} \varrho(\mathbf{x}) \,. \tag{60}$$

By (39) and (45)

$$\det\mathbf{F} = \sqrt{\det(\mathbf{I} + 2\mathbf{E})} + \langle\,\rangle\,, \tag{61}$$

from which, with the aid of the Cayley-Hamilton theory, we find (to first order).

$$\det\mathbf{F} = 1 + \mathrm{tr}\mathbf{E} + \langle\,\rangle \tag{62}$$

We can now obtain the required second local form from (60) with the aid of (46), (49), (51) and (62). Accordingly

$$\varrho(\mathbf{x}) = \varrho_0(\mathbf{x}) - \mathrm{div}(\varrho_0(\mathbf{x})\mathbf{u}) + \langle\,\rangle\,. \tag{63}$$

[Remark: In (63) the notation

$$\mathrm{div}\mathbf{u} = \mathrm{tr}(\partial_{\mathbf{x}}\mathbf{u}) \tag{64}$$

has been used for convenience.]

Balance of momentum: Balance of momentum states that the material time derivative of the momentum will be balanced by the sum of the surface and the body forces, or that

$$\frac{d}{dt}\int_v \mathbf{v}dm = \int_{\partial v} \mathbf{Tn}da + \int_v \mathbf{f}dm + \int_v \underset{s}{\mathbf{F}}dv\,. \tag{65}$$

In (65) $\mathbf{v}$ is the velocity (34), $\mathbf{T} \in L(V_t\,; V_t)$ is (the surface density of) the spatial stress tensor, $\mathbf{n} \in V_t$ is the unit surface normal, and $\mathbf{f} \in V_t$ and $\underset{s}{\mathbf{F}} \in V_t$ are respectively the mass and volume densities of the body force. $\mathbf{f}$ is assumed to be conservative (*i.e.* expressible as the gradient of the gravitational potential). Since $\varrho(\mathbf{x})$ has the form (63), the total gravitational potential can be expressed as the sum of the potential Φ_0 caused by the static part and the potential Φ caused by the dynamical part. In other words

$$\mathbf{f} = \partial_{\mathbf{x}}\Phi_0 + \partial_{\mathbf{x}}\Phi\,. \tag{66}$$

According to Newton's theory of gravitation (see *e.g.* Martensen [25])

$$\Phi_0 = G\int_v \frac{\varrho_0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,dv' \tag{67}$$

and

$$\Phi = -\,G\int_v \frac{\mathrm{div}\,(\varrho_0(\mathbf{x}')\mathbf{u})}{|\mathbf{x} - \mathbf{x}'|}\,dv'\,. \tag{68}$$

In (67) and (68) G is Newton's gravitational constant and $|\mathrm{x} - \mathrm{x}'|$ means the distance between $\mathrm{x} \in E_t$ and $\mathrm{x}' \in E_t$ in the sense of the metric (5). As to Φ, it follows from the convergence of the integral in (68) that Φ is an infinitesimal quantity of first order in $\mathfrak{R}_t$. It may be shown by potential theory methods (see MARTENSEN [25]) that

$$\mathrm{div}\,(\partial_{\mathrm{x}} \Phi_0) = \begin{cases} -\,4\pi G \varrho_0 \,; & \mathrm{x} \in v \\ \quad 0 & ; \mathrm{x} \in v \end{cases} \tag{69}$$

and

$$\mathrm{div}(\partial_{\mathrm{x}} \Phi) = \begin{cases} 4\pi G \mathrm{div}(\varrho_0 \mathrm{u}) \,; & \mathrm{x} \in v \\ \quad 0 & ; \mathrm{x} \in v \end{cases} \tag{70}$$

are the local forms of (67) and (68) respectively.

Since the model used and the numerical method of solution introduce a certain finite number of jump discontinuties for the dynamical variables on the set of surfaces $\{S_i\}$, which are concentric and have no points in common, the application of the ordinary Gauss's integral theorem to (70) in the regions between different S_i, followed by addition, supplies (70) with the jump relation

$$[\partial_{\mathrm{x}} \Phi \cdot \mathbf{n}] = 4\pi G [\varrho_0 \mathrm{u} \cdot \mathbf{n}] \,; \; \mathrm{x} \in \{S_i\} \,, \tag{71}$$

which ought to be satisfied on $\{S_i\}$. The square brackets in (71) mean the jump suffered by the quantity inside.

To make later use of (71) a new variable Ψ will be introduced by

$$\Psi = \partial_{\mathrm{x}} \Phi \cdot \mathbf{n} - 4\pi G \varrho_0 \mathrm{u} \cdot \mathbf{n} \,, \tag{72}$$

Thus the jump condition (71) becomes equivalent to the continuity of Ψ, or

$$[\Psi] = 0 \,; \; \mathrm{x} \in \{S_i\} \,. \tag{73}$$

For future reference it should be noted that the Laplace transformation of (70) and (72) changes only t with p.

Returning to (65) one sees that, before the local form of this equation has been obtained, the surface integral ought to be transformed into a volume integral. Therefore Gauss's integral theorem ought to be modified to take into account the jumps in the dynamical variables on the set of surfaces $\{S_i\}$. For present purposes a sufficiently general modification is found by applying the usual Gauss's integral theorem to the regions between different S_i and adding the results. In this way

$$\int_{\partial v} \mathbf{Tn}da = \int_{v} \operatorname{div}\mathbf{T}\,dv + \sum_i \int_{S_i} [\mathbf{Tn}]da\,, \tag{74}$$

where $[\mathbf{Tn}]$ means the jump in $\mathbf{Tn}$ on S_i.

It follows from the above that the global form (65) of the balance condition for the momentum may, with the aid of (57) and (74), be transformed into

$$\int_{v} (\varrho \frac{d\mathbf{v}}{dt} - \operatorname{div}\mathbf{T} - \varrho\mathbf{f} - \mathbf{F})dv = \sum_i \int_{S_i} [\mathbf{Tn}]da\,, \tag{75}$$

from which

$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div}\mathbf{T} + \varrho\mathbf{f} + \underset{s}{\mathbf{F}} \tag{76}$$

and

$$[\mathbf{Tn}] = 0\,;\ \mathbf{x} \in \{S_i\}\,. \tag{77}$$

(77) merely states that the stress vector $\mathbf{Tn}$ should be continuous on $\{S_i\}$.

In order to obtain the linear infinitesimal form of (76), the various nonlinear terms should be transformed to their first order infinitesimals.

With (47) and (63)

$$\varrho \frac{d\mathbf{v}}{dt} = \varrho_0 \partial_t^2 \mathbf{u} + \langle\,\rangle\,. \tag{78}$$

In *Appendix A* it has been shown that, to first order infinitesimals,

$$\mathbf{T} = p_0\mathbf{I} - \mathbf{u}\cdot\partial_{\mathbf{x}} p_0 \mathbf{I} + \mathbf{T}_D + \langle\,\rangle\,, \tag{79}$$

where $p_0\mathbf{I}$ is the hydrostatic tension and $\mathbf{T}_D$ is the dynamic part of the stress tensor.

By (63) and (66) the term $\varrho\mathbf{f}$ takes the form

$$\varrho\mathbf{f} = \varrho_0 \partial_{\mathbf{x}} \Phi + (\varrho_0 - \operatorname{div}(\varrho_0\mathbf{u}))\partial_{\mathbf{x}} \Phi_0 + \langle\,\rangle \tag{80}$$

The unknown hydrostatic tension in (79) may be eliminated by use of the static equilibrium condition, which is obtained most easily from (76), (79), and (80) by setting the time dependent variables as zero. Thus

$$\partial_{\mathbf{x}} p_0 - \varrho_0(\mathbf{x})\mathbf{g} = 0\,, \tag{81}$$

where $\mathbf{g}$ is the gravitational acceleration defined by

$$\mathbf{g} = -\,\partial_{\mathbf{x}} \Phi_0\,. \tag{82}$$

The Laplace transformed equations of motion of nonrotating media will be obtained from (76) after the substitution of (78), (79), (80), and (81) into it. Thus, to first order infinitesimals,

$$\varrho_0 p^2 \mathbf{u} = -\operatorname{div}(\varrho_0 \mathbf{u} \cdot \mathrm{g}\mathbf{I}) + \operatorname{div}(\varrho_0 \mathbf{u})\mathrm{g} + \operatorname{div}\mathbf{T}_D + \varrho_0 \partial_{\mathrm{x}} \Phi + \underset{s}{\mathbf{F}} + \langle\,\rangle. \qquad (83)$$

The effect of rotation on the balance conditions: In this section certain effects of the observer transformation

$$\mathrm{x}' = \mathbf{Q}\mathrm{x}\,;\, \mathbf{Q}\mathbf{Q}^T = \mathbf{I} \qquad (84)$$
$$t' = t - a$$

will be studied with some care (for a more general account see TRUESDELL and NOLL [40]).

Such vectors and tensors which transform under (84) according to the objective laws

$$\mathbf{u}' = \mathbf{Q}\mathbf{u}$$

and

$$\mathbf{T}' = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$$

do not cause any changes in the form of the balance conditions.

On the other hand, the velocity and the acceleration are not objective. To get their transformation laws (84) is differentiated twice with respect to time. Thus

$$\mathrm{v}' = \frac{d\mathrm{x}'}{dt} - \dot{\mathbf{Q}}\mathbf{Q}^T\mathrm{x}'$$

and

$$\mathrm{a}' = \frac{d^2\mathrm{x}'}{dt^2} - \ddot{\mathbf{Q}}\mathbf{Q}^T\mathrm{x}' - 2\,\dot{\mathbf{Q}}\mathbf{Q}^T\mathrm{v}'\,.$$

Using the results of the previous section, together with the transformation law for the acceleration obtained above and the objectivity of the other vector and tensor fields, the reader may convince himself that the balance condition for the momentum in the rotating coordinate system $(\mathbf{x}')$ is of the form

$$\varrho' \frac{d^2\mathrm{x}'}{dt^2} = \operatorname{div}'\mathbf{T}' + \varrho'\mathrm{f}' + \varrho'\ddot{\mathbf{Q}}\mathbf{Q}^T\mathrm{x}' + 2\,\varrho'\dot{\mathbf{Q}}\mathbf{Q}^T\mathrm{v}' + \underset{s}{\mathbf{F}'}\,. \qquad (85)$$

In order to put the quantity $\dot{\mathbf{Q}}\mathbf{Q}^T\mathbf{v}'$ in a physically more meaningful form, the time derivative of the orthogonality condition

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$$

shows that the tensor

$$\Lambda = \dot{\mathbf{Q}}\mathbf{Q}^T \tag{86}$$

is antisymmetric. In other words

$$\Lambda^T = -\Lambda\,.$$

Therefore there is a vector ω' such that

$$2\Lambda\mathbf{v}' = 2\dot{\mathbf{Q}}\mathbf{Q}^T\mathbf{v}' = 2\omega'\times\mathbf{v}' \tag{87}$$

(see *e.g.* Pach and Frey [27]). If, instead of ω', a new vector ω is introduced by

$$\omega = -\omega'\,, \tag{88}$$

it is seen that ω is readily interpreted as the angular velocity of the system $(\mathbf{x}')$ with respect to the system $(\mathbf{x})$.

The time derivative of (86) with the assumption

$$\dot{\Lambda} = 0$$

(*i.e.* rotation is very weakly time dependent) gives

$$\ddot{\mathbf{Q}}\mathbf{Q}^T\mathbf{x}' = -\dot{\mathbf{Q}}\dot{\mathbf{Q}}^T\mathbf{x}' = -\dot{\mathbf{Q}}\mathbf{Q}^T(\dot{\mathbf{Q}}\mathbf{Q})^T\mathbf{x}' = \Lambda(\Lambda\mathbf{x}')\,.$$

Therefore

$$\ddot{\mathbf{Q}}\mathbf{Q}^T\mathbf{x}' = \omega'\times(\omega'\times\mathbf{x}')\,. \tag{89}$$

It can readily be shown that (89) is the gradient of the scalar field

$$\chi' = -\tfrac{1}{2}|\omega'\times\mathbf{x}'|^2\,.$$

Consequently

$$\ddot{\mathbf{Q}}\mathbf{Q}^T\mathbf{x}' = \partial_{\mathbf{x}'}\chi'\,. \tag{90}$$

As to the Laplace transformed balance condition for the momentum, the substitution of (78), (79), (80), (81), (87), (88), and (90) into (85) results in

$$\begin{aligned}\varrho_0 p^2\mathbf{u} = &-\operatorname{div}(\varrho_0\mathbf{u}\cdot\mathbf{g}\mathbf{I}) + \operatorname{div}(\varrho_0\mathbf{u})\mathbf{g} + \operatorname{div}\mathbf{T}_D\\ &+\varrho_0\partial_{\mathbf{x}}\Phi - 2\varrho_0 p\omega\times\mathbf{u} + \underset{s}{\mathbf{F}} + \langle\,\rangle\,,\end{aligned} \tag{91}$$

to first order infinitesimals.

In (91) the dashes have been dropped and $\mathbf{g}$ is the acceleration due to gravity, defined by

$$\mathbf{g} = -\partial_{\mathbf{x}} \hat{\Phi}_0 ,$$

where $\hat{\Phi}_0$ is the geopotential (see Jeffreys [19]).

Balance of moment of momentum: This condition states that the rate of change of the moment of momentum is balanced by the sum of the moments of the surface tension and of the body forces. This means

$$\frac{d}{dt}\int_v \mathbf{x}\times\mathbf{v}dm = \int_{\partial v} \mathbf{x}\times\mathbf{Tn}da + \int_v \mathbf{x}\times\mathbf{f}dm + \int_v \mathbf{x}\times\underset{s}{\mathbf{F}}dv , \tag{92}$$

from which it follows, with the aid of (57), (74), (76), and (77), that

$$\mathbf{T} = \mathbf{T}^T . \tag{93}$$

In other words the stress tensor ought to be symmetric.

Balance of the energy: This says that the rate of change of the sum of the kinetic and internal energies is balanced by the sum of the mechanical and thermal rates of working, or

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_v \mathbf{v}\,.\,\mathbf{v}dm + \frac{d}{dt}\int_v \varepsilon dm = \int_{\partial v} \mathbf{v}\cdot\mathbf{Tn}da + \int_v \mathbf{v}\cdot\mathbf{f}dm \\ + \int_v \mathbf{v}\cdot\underset{s}{\mathbf{F}}dm + \int_v \mathbf{q}\cdot\mathbf{n}dm + \int_v h dm . \end{aligned} \tag{94}$$

In (94) ε is the mass density of the internal energy, $\mathbf{q}$ is the negative of the heat flux vector, and h is the mass density of the rate of working of the internal heat sources. With the aid of (57), (74) and (76) the energy equation (94) may be put in the forms

$$\varrho\dot{\varepsilon} = \mathrm{tr}(\mathbf{LT}) + \mathrm{div}\mathbf{q} + \varrho h \, ; \; \mathbf{x} \notin \{S_i\} \tag{95}$$

and

$$[\mathbf{v}\cdot\mathbf{Tn}] + [\mathbf{q}\cdot\mathbf{n}] = 0 \, ; \; \mathbf{x} \in \{S_i\} . \tag{96}$$

In (95) $\mathbf{L}$ is the velocity gradient defined in (36). As to the jump relation (96), it is seen that, using the continuity of $\mathbf{v}$ and the relation (77), (96) reduces to

$$[\mathbf{q} \cdot \mathbf{n}] = 0 \; ; \; \mathbf{x} \in \{S_i\} \, . \tag{97}$$

A more useful form of the energy equation (95) will be obtained, if, instead of $\mathbf{T}$, use is made of the Piola stress tensor $\mathbf{T}^*$ (see *e.g.* COLEMAN [8]), which is defined by

$$\mathbf{T}^* = \frac{1}{\varrho} \, \mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{T^{-1}} \, . \tag{98}$$

Hence $\mathbf{T}^*$ together with

$$\mathrm{tr}(\dot{\mathbf{F}}\mathbf{T}^*\mathbf{F}^T) = \mathrm{tr}(\dot{\mathbf{F}}^T\mathbf{F}\mathbf{T}^*)$$

gives, instead of (95),

$$\varrho\dot{\varepsilon} = \frac{\varrho}{2} \, \mathrm{tr}(\mathbf{T}^*\dot{\mathbf{C}}) + \mathrm{div}\mathbf{q} + \varrho h \, . \tag{99}$$

In *Appendix B* it has been shown that the Laplace transformed equation of energy has the forms (100) and (102) in isotropic and in transversely isotropic symmetry classes respectively.

— Isotropic symmetry:

$$\mathrm{div}\mathbf{q} = \varrho_0 p \tilde{c}_E T + \vartheta_0 p \underset{0}{\tilde{\gamma}} \mathrm{div}\mathbf{u} + \langle \, \rangle \, , \tag{100}$$

where, corresponding to the three memory models used in *Appendix A*, the parameters $\tilde{c}_E$ and $\underset{0}{\tilde{\gamma}}$ have the following values (see further *Appendix B*):

$$\begin{aligned}
&\text{Model 1:} \quad && \tilde{c}_E = c_E \, , && \underset{0}{\tilde{\gamma}} = \gamma \\
&\text{Model 2:} \quad && \tilde{c}_E = c_E + \sum_{i=1}^{l} c_i p^i \, , && \underset{0}{\tilde{\gamma}} = \gamma + \sum_{i=1}^{k} \hat{\gamma}_i p^i \\
&\text{Model 3:} \quad && \tilde{c}_E = c_E + \sum_{i=1}^{l} c_i p^i \, , && \underset{0}{\tilde{\gamma}} = \gamma + \sum_{i=1}^{k} \hat{\gamma}_i p^i
\end{aligned} \tag{101}$$

— Transversely isotropic symmetry:

$$\mathrm{div}\mathbf{q} = \varrho_0 p \tilde{c}_E T + \vartheta_0 p \underset{0}{\tilde{\gamma}} \mathrm{div}\mathbf{u} + \vartheta_0 p \underset{0}{\tilde{\gamma}}' u_{r,r} + \langle \, \rangle \, . \tag{102}$$

According to *Appendix B* the parameters in the three memory models used are given by

$$\begin{aligned}
&\text{Model 1:} \quad \tilde{c}_E = c_E \quad , \; \tilde{\gamma}_0 = \gamma \quad , \; \tilde{\gamma}'_0 = \gamma' \\
&\text{Model 2:} \quad \tilde{c}_E = c_E + \sum_{i=1}^{l} c_i p^i \, , \; \tilde{\gamma}_0 = \gamma + \sum_{i=1}^{k} \hat{\gamma}_i p^i \, , \; \tilde{\gamma}'_0 = \gamma' + \sum_{i=1}^{k} \hat{\gamma}'_i p^i \qquad (103) \\
&\text{Model 3:} \quad \tilde{c}_E = c_E + \sum_{i=1}^{l} c_i p^i \, , \; \tilde{\gamma}_0 = \gamma + \sum_{i=1}^{k} \hat{\gamma}_i p^i \, , \; \tilde{\gamma}'_0 = \gamma' + \sum_{i=1}^{k} \hat{\gamma}'_i p^i
\end{aligned}$$

Clausius-Duhem inequality: This expresses the fact that the production of the entropy is a non-negative quantity, or that

$$\frac{d}{dt} \int_v \eta dm - \int_{\partial v} \frac{\mathbf{q} \, . \, \mathbf{n}}{\vartheta} da - \int_v \frac{h}{\vartheta} dm \geq 0 \, , \qquad (104)$$

where η is the mass density of the entropy and ϑ is the absolute temperature. However, before the local form of (104) is given, the new variables

$$\mathbf{h} = \partial_X \vartheta \, , \qquad (105)$$

$$\mathbf{q}^* = \frac{1}{\varrho} \mathbf{F}^{-1} \mathbf{q} \, , \qquad (106)$$

and

$$\psi = \varepsilon - \vartheta \eta \qquad (107)$$

are introduced.

Next, the use of (57), (74) (applied to $\mathbf{q}$), (97), (99), (105), (106) and (107), together with

$$\frac{\mathbf{q} \cdot \partial_X \vartheta}{\varrho \vartheta} = \frac{\mathbf{h} \cdot \mathbf{F}^{-1} \mathbf{q}}{\varrho \vartheta} = \frac{\mathbf{h} \cdot \mathbf{q}^*}{\vartheta} \, ,$$

gives, instead of (104),

$$\tfrac{1}{2} \operatorname{tr}(\mathbf{T}^* \dot{\mathbf{C}}) - \dot{\psi} - \dot{\vartheta} \eta + \frac{\mathbf{h} \cdot \mathbf{q}^*}{\vartheta} \geq 0 \, . \qquad (108)$$

Constitutive equations: A study of the equations (70), (72), (83) (or (91)) and (100) (or (102)) shows that the dynamic variables form a $3 + 1 + 1 + 6 + 3 + 1 = 15$ dimensional product space, while the range of the equations for the dynamic variables in question is only $1 + 1 + 3 + 1 = 6$ dimensional. Because of this high underdeterminacy of the previous system of equations, additional relations between the dynamic variables

$\mathbf{u}$, $\mathbf{T}_D$, Φ, ψ, $\mathbf{q}$, and T are needed to fill the lacking 9 dimensions. Such relations are traditionally called constitutive relations.

Instead of using the classical examples (Hooke's stress-strain relation and Fourier's heat conduction law) the purpose of this paper is to place the constitutive theory on the axiomatic basis of modern continuum mechanics. A proper set of axioms for the present purposes is found in the works by TRUESDELL and NOLL [40], ERINGEN [11], and JAUNZEMIS [16]. These axioms are:

A. Causality: In a thermomechanical system the independent variables are the deformation and the temperature.
B. Determinism: The deformation and the temperature are influenced by the previous history in space and time. The history in space means that the response at the spatial point $\mathbf{x}$ is influenced by the deformations of all material points of the body, whereas the history in time means that the response at $\mathbf{x}$ is influenced by all times prior to the present one.
C. Equipresence: Every constitutive equation has *a priori* the same list of independent variables provided that this does not contradict other principles of continuum mechanics (*e.g.* balance conditions and other axioms).
D. Objectivity: The response of the material is invariant under the rotations of the spatial frame of reference $\mathbf{x}$.
E. Material invariance: The response of the material is invariant under some subgroups of the full rotation group of the material frame of reference $\mathbf{X}$.
F. Local action: Only a small neighbourhood of the material point $\mathbf{X}$ should influence the response at $\mathbf{x}$.
G. Memory: The memory with respect to the past histories in time should be »short» (i.e. except in a small neighbourhood of the present time all other times from the past should be neglected).
H. Admissibility: The constitutive equations should not contradict the other principles of continuum mechanics (e.g. the Clausius-Duhem inequality (108)).

If the axiom of memory (G) is satisfied by a certain finite number of the time derivatives of the independent variables ($\mathbf{C}$, ϑ, $\mathbf{h}$), *Appendix A* shows that the Laplace transformed stress-strain and heat conduction relations, in the two symmetry classes (isotropic and transversely isotropic), take on the following forms:

— Isotropic symmetry:

$$\mathbf{T}_D = -\tilde{\gamma}T\mathbf{I} + \tilde{\lambda}\,\mathrm{tr}\,\mathbf{e}\mathbf{I} + 2\tilde{\mu}\mathbf{e} + \langle\,\rangle \tag{109}$$

and

$$\mathbf{q} = \tilde{\varkappa}\partial_{\mathbf{x}}T + \langle\,\rangle\,. \tag{110}$$

In (109) $\tilde{\gamma}$ is the Laplace transformed stress temperature modulus, $\tilde{\lambda}$ and $\tilde{\mu}$ are the Laplace transformed stress-strain moduli. In (110) $\tilde{\varkappa}$ is in turn the Laplace transformed heat conduction modulus.

As to the explicit forms of these moduli, *Appendix A* shows that, in the three memory models studied here, we obtain:

Model 1: $$\tilde{\lambda} = \lambda - p_0 + \sum_{i=1}^{k} \lambda_i p^i\,, \quad \tilde{\mu} = \mu + p_0 + \sum_{i=1}^{k} \mu_i p^i\,,$$

$$\tilde{\gamma} = \gamma \qquad\qquad , \quad \tilde{\varkappa} = \varkappa$$

Model 2: $$\tilde{\lambda} = \lambda - p_0 + \sum_{i=1}^{k} \lambda_i p^i\,, \quad \tilde{\mu} = \mu + p_0 + \sum_{i=1}^{k} \mu_i p^i\,,$$

$$\tilde{\gamma} = \gamma + \sum_{i=1}^{l} \gamma_i p^i\,, \quad \tilde{\varkappa} = \varkappa$$

Model 3: $$\tilde{\lambda} = \lambda - p_0 + \sum_{i=1}^{k} \lambda_i p^i\,, \quad \tilde{\mu} = \mu + p_0 + \sum_{i=1}^{k} \mu_i p^i\,,$$

$$\tilde{\gamma} = \gamma + \sum_{i=1}^{l} \gamma_i p^i\,, \quad \tilde{\varkappa} = \varkappa + \sum_{i=1}^{m} \varkappa_i p^i$$

— Transversely isotropic symmetry:

$$\mathbf{T}_D = -(\tilde{\gamma}\mathbf{I} + \mathbf{x}^0 \otimes \mathbf{x}^0\tilde{\gamma}')T + \mathbf{x}^0 \otimes \mathbf{x}^0(\tilde{\beta}e_{rr} + \tilde{\lambda}e^{\gamma}_{\gamma}) + (\mathbf{x}^0 \otimes \mathbf{a}_{\alpha} + \mathbf{a}_{\alpha} \otimes \mathbf{x}^0)\,2\tilde{\mu}e_{r\alpha} + \mathbf{a}_{\alpha} \otimes \mathbf{a}_{\beta}[a^{\alpha\beta}(\tilde{\lambda}'e^{\gamma}_{\gamma} + \tilde{\lambda}e_{rr}) + 2\tilde{\mu}'e^{\alpha\beta}] + \langle\,\rangle \tag{111}$$

and

$$\mathbf{q} = (\tilde{\varkappa}\mathbf{I} + \mathbf{x}^0 \otimes \mathbf{x}^0\tilde{\varkappa}')\,\partial_{\mathbf{x}}T + \langle\,\rangle\,. \tag{112}$$

As seen above, in this symmetry class there are two stress temperature moduli $\tilde{\gamma}$ and $\tilde{\gamma}'$, five stress strain moduli $\tilde{\beta}$, $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\lambda}'$ and $\tilde{\mu}'$, and two heat conduction moduli $\tilde{\varkappa}$ and $\tilde{\varkappa}'$. For the explicit expressions for these moduli *Appendix A* shows that

Model 1:

$$\tilde{\beta} = \beta + p_0 + \sum_{i=1}^{k} \beta_i p^i\,, \quad \tilde{\lambda} = \lambda - p_0 + \sum_{i=1}^{k} \lambda_i p^i\,, \quad \tilde{\lambda}' = \lambda' - p_0 + \sum_{i=1}^{k} \lambda_i' p^i$$

$$\tilde{\mu} = \mu + p_0 + \sum_{i=1}^{k} \mu_i p^i\,, \quad \tilde{\mu}' = \mu' + p_0 + \sum_{i=1}^{k} \mu_i' p^i\,, \quad \tilde{\gamma} = \gamma$$

$$\tilde{\gamma}' = \gamma'\,, \quad \tilde{\varkappa} = \varkappa\,, \quad \tilde{\varkappa}' = \varkappa$$

Model 2:

$$\tilde{\beta} = \beta + p_0 + \sum_{i=1}^{k} \beta_i p^i\,, \quad \tilde{\lambda} = \lambda - p_0 + \sum_{i=1}^{k} \lambda_i p^i\,, \quad \tilde{\lambda}' = \lambda' - p_0 + \sum_{i=1}^{k} \lambda_i' p^i$$

$$\tilde{\mu} = \mu + p_0 + \sum_{i=1}^{k} \mu_i p^i\,, \quad \tilde{\mu}' = \mu' + p_0 + \sum_{i=1}^{k} \mu_i' p^i\,, \quad \tilde{\gamma} = \gamma + \sum_{i=1}^{l} \gamma_i p^i$$

$$\tilde{\gamma}' = \gamma' + \sum_{i=1}^{l} \gamma_i' p^i\,, \quad \tilde{\varkappa} = \varkappa\,, \quad \tilde{\varkappa}' = \varkappa'$$

Model 3:

$$\beta = \beta + p_0 + \sum_{i=1}^{k} \beta_i p^i\,, \quad \tilde{\lambda} = \lambda - p_0 + \sum_{i=1}^{k} \lambda_i p^i\,, \quad \tilde{\lambda}' = \lambda' - p_0 + \sum_{i=1}^{k} \lambda_i' p^i$$

$$\tilde{\mu} = \mu + p_0 + \sum_{i=1}^{k} \mu_i p^i\,, \quad \tilde{\mu}' = \mu' + p_0 + \sum_{i=1}^{k} \mu_i' p^i\,, \quad \tilde{\gamma} = \gamma + \sum_{i=1}^{l} \gamma_i p^i$$

$$\tilde{\gamma}' = \gamma' + \sum_{i=1}^{l} \gamma_i' p^i\,, \quad \tilde{\varkappa} = \varkappa + \sum_{i=1}^{m} \varkappa_i p^i\,, \quad \tilde{\varkappa}' = \varkappa' + \sum_{i=1}^{m} \varkappa_i' p^i$$

Equations of motion in a rotating coordinate system

Since the range of the equations (109) and (110) (or (111) and (112)) is 9 dimensional, it follows that the system (70), (72), (91), (100) (or (102)), (109) (or (111)), and (110) (or (112)) is complete with respect to the dynamic variables. Consequently the Laplace transformed equations of motion for the rotating thermoviscoelastic sphere in the two symmetry classes assume the following forms:

— Isotropic symmetry:

$$\varrho_0 p^2 \mathbf{u} = -\operatorname{div}(\varrho_0 \mathbf{u} \cdot \mathbf{g} \mathbf{I}) + \operatorname{div}(\varrho_0 \mathbf{u}) \mathbf{g} + \operatorname{div} \mathbf{T}_D + \varrho_0 \partial_{\mathbf{x}} \Phi - 2 \varrho_0 p \boldsymbol{\omega} \times \mathbf{u} + \underset{s}{\mathbf{F}} + \langle\,\rangle\,, \tag{113}$$

$$\mathbf{T}_D = -\tilde{\gamma} T \mathbf{I} + \tilde{\lambda} \operatorname{tr} \mathbf{e}\, \mathbf{I} + 2 \tilde{\mu} \mathbf{e} + \langle\,\rangle\,, \tag{114}$$

$$\mathrm{div}\mathbf{q} = \varrho_0 p\tilde{c}_E T + \vartheta_0 p\underset{0}{\tilde{\gamma}}\mathrm{div}\mathbf{u} + \langle\,\rangle\,, \tag{115}$$

$$\mathbf{q} = \tilde{\varkappa}\partial_{\mathbf{x}} T + \langle\,\rangle\,, \tag{116}$$

$$\mathrm{div}(\partial_{\mathbf{x}}\Phi) = 4\pi G\mathrm{div}(\varrho_0\mathbf{u}) + \langle\,\rangle\,, \tag{117}$$

$$\Psi = \frac{\partial\Phi}{\partial n} - 4\pi G\varrho_0\mathbf{u}\cdot\mathbf{n}\,, \tag{118}$$

$$\mathrm{div}\mathbf{g} = 4\pi G\varrho_0 \tag{119}$$

— Transversely isotropic symmetry:

In this symmetry class the equations of motion are the same as before, except that (113), (114) and (115) ought to be substituted by

$$\mathbf{T}_D = -(\tilde{\gamma}\mathbf{I} + \mathbf{x}^0\otimes\mathbf{x}^0\tilde{\gamma}')T + \mathbf{x}^0\otimes\mathbf{x}^0(\tilde{\beta}e_{rr} + \tilde{\lambda}e^{\gamma}_{\gamma}) + (\mathbf{x}^0\otimes\mathbf{a}_\alpha \tag{114*}$$
$$+ \mathbf{a}_\alpha\otimes\mathbf{x}^0)2\tilde{\mu}e^{r\alpha} + \mathbf{a}_\alpha\otimes\mathbf{a}_\beta[a^{\alpha\beta}(\tilde{\lambda}'e^{\gamma}_{\gamma} + \tilde{\lambda}e_{rr}) + 2\tilde{\mu}'e^{\alpha\beta}] + \langle\,\rangle\,,$$

$$\mathrm{div}\mathbf{q} = \tilde{\varrho}_0 p\tilde{c}_E T + \vartheta_0 p\underset{0}{\tilde{\gamma}}\mathrm{div}\mathbf{u} + \vartheta_0 p\underset{0}{\tilde{\gamma}}'u_{r,r} + \langle\,\rangle\,, \tag{115*}$$

$$\mathbf{q} = \tilde{\varkappa}\partial_{\mathbf{x}}T + \tilde{\varkappa}'T_{,r}\mathbf{x}^0 + \langle\,\rangle\,. \tag{116*}$$

Transformation of the previous equations of motion into scalar systems: Transformation of the equations (113)—(119) into sets of scalar equations may be accomplished in the spherical symmetry case by using a representation theorem for tangent (surface) vector fields (see BACKUS [4] or [5]). Accordingly, any tangent vector field $\mathbf{a}$ (a vector field with no normal component) may be represented on the unit sphere with the aid of two scalar fields A and B as

$$\mathbf{a} = \mathbf{a}_\alpha(a^{\alpha\beta}A_{|\beta} + \varepsilon^{\alpha\beta}B_{|\beta})\,, \tag{120}$$

where $\{\mathbf{a}_\alpha\}$ is the base system (28) on the unit sphere S_1, $a^{\alpha\beta}$ are the contravariant components of the surface metric tensor, and $\varepsilon^{\alpha\beta}$ are the contravariant components of the surface rotator $-\mathbf{x}_0\times\mathbf{I}$.

The second step in the transformation is to express the radial components of every space vector, as well as the scalar potentials of its surface part (120), in a series of spherical surface harmonics. Following GOERTZEL and TRALLI [14] the surface harmonics adopted here are the eigenfunctions of the systems below, *i.e.:*

$$\nabla_s^2 X_l^m = -l(l+1)X_l^m \tag{121}$$

and

$$\partial_\varphi X_l^m = imX_l^m ,$$

where

$$\nabla_s^2 = \nabla_s \cdot \nabla_s \tag{122}$$

and

$$\nabla_s = \mathbf{a}^\gamma \partial_\gamma = \Theta^0 \partial_\Theta + \frac{\varphi^0}{\sin\Theta}\,\partial_\varphi .$$

The last expression in (122) has been obtained by changing the base system $\{\mathbf{a}_\gamma\}$ into the orthonormal system (Θ^0, φ^0). As to the eigenfunctions, it is seen that they form an orthonormal system with respect to both the indices over the surface of the unit sphere S_1. In other words

$$\int_{S_1} \overset{*}{X}{}_{l'}^{m'} X_l^m da = \delta_{m'm}\delta_{l'l} , \tag{123}$$

where $\overset{*}{X}{}_{l'}^{m'}$ means the complex conjugate of $X_{l'}^{m'}$ and

$$\delta_{mm'} = \begin{cases} 1\,; & m = m' \\ 0\,; & m \neq m' . \end{cases}$$

According to the previous discussion the dynamical field quantities T, Φ, $\mathbf{u}$, $\mathbf{q}$, $\underset{s}{\mathbf{F}}$, $\omega\times\mathbf{u}$, and $\mathbf{T}_D$ are expressed as follows:

$$T = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} T_l X_l^m , \tag{124}$$

$$\Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Phi_l X_l^m , \tag{125}$$

$$\mathbf{u} = \sum_{l=0}^{\infty} \sum_{m=-l} [U_l \mathbf{x}^0 X_l^m + \mathbf{a}_\alpha (V_l a^{\alpha\beta} X_{l|\beta}^m + W_l \varepsilon^{\alpha\beta} X_{l|\beta}^m)] , \tag{126}$$

$$\mathbf{q} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [Q_l \mathbf{x}^0 X_l^m + \mathbf{a}_\alpha (M_l a^{\alpha\beta} X_{l|\beta}^m + N_l \varepsilon^{\alpha\beta} X_{l|\beta}^m)] , \tag{127}$$

$$\underset{s}{\mathbf{F}} = \sum_{l=0}^{\infty} \sum_{m=-l} [\underset{r}{F_l} \mathbf{x}^0 X_l^m + \mathbf{a}_\alpha (\underset{a}{F_l} a^{\alpha\beta} X_{l|\beta}^m + \underset{s}{F_l} \varepsilon^{\alpha\beta} X_{l|\beta}^m)] , \tag{128}$$

$$\boldsymbol{\omega}\times\mathbf{u} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}[\hat{U}_l\mathbf{x}^0X_l^m + \mathbf{a}_\alpha(\hat{V}_l a^{\alpha\beta}X_{l|\beta}^m + \hat{W}_l\varepsilon^{\alpha\beta}X_{l|\beta}^m)] , \tag{129}$$

$$\mathbf{T}_D = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}[P_l\mathbf{x}^0 \otimes \mathbf{x}^0 X_l^m + (\mathbf{x}^0 \otimes \mathbf{a}_\alpha + \mathbf{a}_\alpha \otimes \mathbf{x}^0)(R_l a^{\alpha\beta}X_{l|\beta}^m \tag{130}$$

$$+ S_l\varepsilon^{\alpha\beta}X_{l|\beta}^m) + T^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta X_l^m] .$$

Because of the length of the calculations involved in the substitution of (124)—(130) into both the systems of Laplace transformed equations of motion given in (113)—(119), the details have been left out. (The interested reader is referred to the excellent article by Backus [5], where the scalar decomposition of vector and tensor fields has been treated in detail.)

As the result of the above process the systems (113)—(119) are decomposed in each symmetry class into two coupled systems of ordinary differential equations (131), which describe (with respect to the surface harmonics) the coefficients of the spheroidal (poloidal) and toroidal (torsional) oscillations of the rotating model earth.

In terms of matrix calculus the systems above are of the form

$$\frac{dy}{dr} = A(p\,;r)y - 2\,\varrho_0 p\omega \underset{d}{f} - \underset{s}{f} , \tag{131}$$

where ω is the component of the angular velocity vector

$$\boldsymbol{\omega} = \omega\mathbf{k} \tag{132}$$

along the axis of rotation $\mathbf{k}$ (in this work $\mathbf{k}$ is the unit vector on the z-axis of the Cartesian coordinate system embedded in E_t). It is obvious from the discussion above that the vectors $y, \underset{d}{f}, \underset{s}{f}$ and the elements of the matrix $A(p\,;r)$ have different expressions according to the material symmetry (isotropic or transversely isotropic) and according to the type of oscillation they are permitted to do (spheroidal or toroidal). Consequently we find

— Isotropic symmetry:

1. Spheroidal oscillations:

$$y = \begin{vmatrix} U_l \\ V_l \\ T_l \\ \Phi_l \\ P_l \\ R_l \\ Q_l \\ \Psi_l \end{vmatrix}, \quad \underset{d}{f} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \hat{U}_l \\ \hat{V}_l \\ 0 \\ 0 \end{vmatrix}, \quad \underset{s}{f} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \underset{r}{F_l} \\ \underset{a}{F_l} \\ 0 \\ 0 \end{vmatrix} \tag{133}$$

and

$$A_{11} = -\frac{2\tilde{\lambda}}{r(\tilde{\lambda} + 2\tilde{\mu})}, \quad A_{12} = \frac{l(l+1)\tilde{\lambda}}{r(\tilde{\lambda} + 2\tilde{\mu})},$$

$$A_{13} = \frac{\tilde{\gamma}}{\tilde{\lambda} + 2\tilde{\mu}}, \quad A_{15} = \frac{1}{\tilde{\lambda} + 2\tilde{\mu}}$$

$$A_{21} = -\frac{1}{r}, \quad A_{22} = \frac{1}{r}, \quad A_{26} = \frac{1}{\tilde{\mu}}$$

$$A_{37} = \frac{1}{\tilde{\varkappa}}$$

$$A_{41} = 4\pi\varrho_0 G, \quad A_{48} = 1$$

$$A_{51} = \varrho_0 p^2 - \frac{4\varrho_0 g}{r} + \frac{4\tilde{\mu}(2\tilde{\mu} + 3\tilde{\lambda})}{r^2(\tilde{\lambda} + 2\tilde{\mu})}, \tag{134}$$

$$A_{52} = l(l+1)\left[\frac{\varrho_0 g}{r} - \frac{2\tilde{\mu}(2\tilde{\mu} + 3\tilde{\lambda})}{r^2(\tilde{\lambda} + 2\tilde{\mu})}\right]$$

$$A_{53} = -\frac{4\tilde{\gamma}\tilde{\mu}}{r(\tilde{\lambda} + 2\tilde{\mu})}, \quad A_{55} = -\frac{4\tilde{\mu}}{r(\tilde{\lambda} + 2\tilde{\mu})}, \quad A_{56} = \frac{l(l+1)}{r}, \quad A_{58} = -\varrho_0$$

$$A_{61} = \frac{\varrho_0 g}{r} - \frac{2\tilde{\mu}(2\tilde{\mu} + 3\tilde{\lambda})}{r^2(\tilde{\lambda} + 2\tilde{\mu})}, \quad A_{62} = \varrho_0 p^2 - \frac{2\tilde{\mu}}{r^2} + \frac{4l(l+1)\tilde{\mu}(\tilde{\lambda} + \tilde{\mu})}{r^2(\tilde{\lambda} + 2\tilde{\mu})}$$

$$A_{63} = \frac{2\tilde{\gamma}\tilde{\mu}}{r(\tilde{\lambda} + 2\tilde{\mu})}, \quad A_{64} = -\frac{\varrho_0}{r}, \quad A_{65} = -\frac{\tilde{\lambda}}{r(\tilde{\lambda} + 2\tilde{\mu})}, \quad A_{66} = -\frac{3}{r}$$

$$A_{71} = \frac{4\,\vartheta_0 \tilde{\mu} \underset{0}{\tilde{\gamma}} p}{r(\tilde{\lambda} + 2\tilde{\mu})}, \quad A_{72} = -\frac{2\,l(l+1)\vartheta_0 \tilde{\mu} \underset{0}{\tilde{\gamma}} p}{r(\tilde{\lambda} + 2\tilde{\mu})}$$

$$A_{73} = \varrho_0 \tilde{c}_E p + \frac{l(l+1)\tilde{\varkappa}}{r^2} + \frac{\vartheta_0 \tilde{\gamma} \underset{0}{\tilde{\gamma}} p}{\tilde{\lambda} + 2\tilde{\mu}}, \quad A_{75} = \frac{\vartheta_0 \underset{0}{\tilde{\gamma}} p}{\tilde{\lambda} + 2\tilde{\mu}}, \quad A_{77} = -\frac{2}{r}$$

$$A_{82} = -\frac{4\,l(l+1)\pi\varrho_0 G}{r}, \quad A_{84} = \frac{l(l+1)}{r^2}, \quad A_{88} = -\frac{2}{r}.$$

2. Toroidal oscillations:

$$y = \begin{pmatrix} W_l \\ S_l \end{pmatrix}, \quad \underset{d}{f} = \begin{pmatrix} 0 \\ \hat{W}_l \end{pmatrix}, \quad \underset{s}{f} = \begin{pmatrix} 0 \\ \underset{s}{F_l} \end{pmatrix} \tag{135}$$

and

$$A_{11} = \frac{1}{r}, \quad A_{12} = \frac{1}{\tilde{\mu}}$$

$$A_{21} = \varrho_0 p^2 + \frac{(l^2 + l - 2)\tilde{\mu}}{r^2}, \quad A_{22} = -\frac{3}{r}. \tag{136}$$

— Transversely isotropic symmetry:

1. Spheroidal oscillations:

$y, \underset{d}{f}$ and $\underset{s}{f}$ have the expressions (133) and

$$A_{11} = -\frac{2\tilde{\lambda}}{r\tilde{\beta}}, \quad A_{12} = \frac{l(l+1)\tilde{\lambda}}{r\tilde{\beta}}, \quad A_{13} = \frac{\tilde{\gamma} + \tilde{\gamma}'}{\tilde{\beta}}, \quad A_{15} = \frac{1}{\tilde{\beta}}$$

$$A_{21} = -\frac{1}{r}, \quad A_{22} = \frac{1}{r}, \quad A_{26} = \frac{1}{\tilde{\mu}}$$

$$A_{37} = \frac{1}{\tilde{\varkappa} + \tilde{\varkappa}'} \tag{137}$$

$$A_{41} = 4\pi\varrho_0 G, \quad A_{48} = 1$$

$$A_{51} = \varrho_0 p^2 - \frac{4\varrho_0 g}{r} + \frac{4(\tilde{\lambda}' + \tilde{\mu}')}{r^2} - \frac{4\tilde{\lambda}^2}{r^2\tilde{\beta}},$$

$$A_{52} = l(l+1)\left[\frac{\varrho_0 g}{r} - \frac{2(\tilde\lambda' + \tilde\mu')}{r^2} + \frac{2\,\tilde\lambda^2}{r^2\tilde\beta}\right]$$

$$A_{53} = \frac{2\,\tilde\lambda(\tilde\gamma + \tilde\gamma')}{r\tilde\beta} - \frac{2\,\tilde\gamma}{r}\,, \quad A_{55} = \frac{2}{r}\left(\frac{\tilde\lambda}{\tilde\beta} - 1\right),$$

$$A_{56} = \frac{l(l+1)}{r}\,, \quad A_{58} = -\,\varrho_0$$

$$A_{61} = \frac{\varrho_0 g}{r} - \frac{2(\tilde\lambda' + \tilde\mu')}{r^2} + \frac{2\,\tilde\lambda^2}{r^2\tilde\beta}\,, \quad A_{62} = \varrho_0 p^2$$

$$+ \frac{l(l+1)(\tilde\lambda' + 2\tilde\mu')}{r^2} - \frac{l(l+1)\tilde\lambda^2}{r^2\tilde\beta} - \frac{2\,\tilde\mu'}{r^2}$$

$$A_{63} = \frac{\tilde\gamma}{r} - \frac{\tilde\lambda(\tilde\gamma + \tilde\gamma')}{r\tilde\beta}\,, \quad A_{64} = -\,\frac{\varrho_0}{r}\,, \quad A_{65} = -\,\frac{\lambda}{r\tilde\beta}\,, \quad A_{66} = -\,\frac{3}{r}$$

$$A_{71} = \frac{2\vartheta_0 \underset{0}{\tilde\gamma} p}{r} - \frac{2\,\tilde\lambda\vartheta_0 p(\underset{0}{\tilde\gamma} + \underset{0}{\tilde\gamma'})}{r\tilde\beta}\,,$$

$$A_{72} = l(l+1)\left[\frac{\tilde\lambda\vartheta_0 p(\underset{0}{\tilde\gamma} + \underset{0}{\tilde\gamma'})}{r\tilde\beta} - \frac{\vartheta_0 \underset{0}{\tilde\gamma} p}{r}\right]$$

$$A_{73} = \varrho_0 \tilde c_E p + \frac{l(l+1)\tilde\varkappa}{r^2} + \frac{\vartheta_0 p(\underset{0}{\tilde\gamma} + \underset{0}{\tilde\gamma'})(\tilde\gamma + \tilde\gamma')}{\tilde\beta}\,,$$

$$A_{75} = \frac{\vartheta_0 p(\underset{0}{\tilde\gamma} + \underset{0}{\tilde\gamma'})}{\tilde\beta}\,, \quad A_{77} = -\,\frac{2}{r}$$

$$A_{82} = -\,\frac{4l(l+1)\pi\varrho_0 G}{r}\,, \quad A_{84} = \frac{l(l+1)}{r^2}\,, \quad A_{88} = -\,\frac{2}{r}$$

2. Toroidal oscillations:

y, $\underset{d}{f}$ and $\underset{s}{f}$ are the same as in (135) and

$$A_{11} = \frac{1}{r}\,, \quad A_{12} = \frac{1}{\tilde\mu}$$

$$A_{21} = \varrho_0 p^2 + \frac{(l^2 + l - 2)\tilde\mu'}{r^2}\,, \quad A_{22} = -\,\frac{3}{r}\,. \tag{138}$$

Boundary conditions for spheroidal and toroidal oscillations: Because of the parameter p in the matrices (134), (136), (137), and (138) and in the vectors $\underset{d}{f}$ and $\underset{s}{f}$, the introduction of certain boundary conditions makes it possible to solve the coefficients U_l, V_l and W_l as a result of certain eigenfunction expansions.

Guided by physical intuition concerning the seismic model adopted, we expect that the penetration depth of the seismic energy varies greatly for different parts of the eigenvalue spectrum. Thus, given a certain bandwidth of eigenfrequencies (discrete in case of the free oscillations), there is a depth $(r < r_0)$ below which the medium is practically in a state of static equilibrium. Another set of boundary conditions arises naturally from the behaviour of the dynamic variables on the free surface of the model earth $(r = a)$.

Since there is a good deal of seismic literature concerning the boundary conditions for U_l, V_l, Φ_l, P_l, R_l, Ψ_l, W_l, and S_l (see *e.g.* Alterman *et. al.* [2] and TAKEUCHI and SAITO [38]), only the boundary conditions for T_l and Q_l need some consideration here. According to what has been said about the vanishing of the seismic energy at r_0, T_l ought to vanish there. As to the radial component Q_l of the heat conduction vector, it is seen to vanish at the free surface $r = a$.

In matrix notation the boundary conditions described above take the form

$$W_0 y_0 + W_a y_a = 0 \, , \tag{139}$$

where the subscripts o and a refer to $r = r_0$ and $r = a$ respectively. The matrices W_0 and W_a are for spheroidal and toroidal oscillations respectively as follows.

— Spheroidal oscillations:

$$W_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad W_a = \begin{pmatrix} 0 & 0 \\ W & I \end{pmatrix} , \tag{140}$$

where I and 0 are the 4×4 unit and null matrices, respectively, and W is the matrix

$$W = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{l+1}{a} \end{vmatrix} . \tag{141}$$

— Toroidal oscillations:

$$W_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad W_a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{142}$$

Boundary conditions for tidal deformations: Since the coefficients with respect to the surface harmonics of the displacement vector $\mathbf{u}$ and of the gravity potential Φ in tidal deformations are obtained in the next chapter, it is relevant in this connection to give the boundary conditions needed.

The boundary conditions given by TAKEUCHI and SAITO [38] added to the conditions for T_l and Q_l obtained in the previous section are in matrix notation of the form

$$W_0 y_0 + W_a y_a = b\,, \tag{143}$$

where the boundary matrices have been given in (140) and b is the column vector

$$b = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{2l+1}{a} \end{Vmatrix}. \tag{144}$$

Body force expressed by the jumps in the displacement and in the normal stress vectors: Occasionally, instead of a volume density $(\underset{s}{\mathbf{F}})$, the focus mechanism is expressed by the jump discontinuity in the displacement vector and in the normal stress vector (see *e.g.* ALTERMAN *et. al.* [2] and SAITO [33]). In other words

$$[\mathbf{u}] = \underset{s}{\mathbf{u}} \quad \text{and} \quad [\mathbf{T}_D \mathbf{x}^0] = \underset{s}{\mathbf{t}}\,, \quad (r = r_s)\,, \tag{145}$$

where $\underset{s}{\mathbf{u}}$ and $\underset{s}{\mathbf{t}} \in V_t$ describe the magnitude of the focus on the sphere $r = r_s$.

To transform the above jump conditions into scalar form use will be made of the decompositions (126) and (130) together with

$$\underset{s}{\mathbf{u}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [\underset{s}{U_l} \mathbf{x}^0 X_l^m + \mathbf{a}_\alpha(\underset{s}{V_l} a^{\alpha\beta} X_{l|\beta}^m + \underset{s}{W_l} \varepsilon^{\alpha\beta} X_{l|\beta}^m)] \tag{146}$$

and

$$\underset{s}{\mathbf{t}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [\underset{s}{P_l} \mathbf{x}^0 X_l^m + \mathbf{a}_\alpha(\underset{s}{R_l} a^{\alpha\beta} X_{l|\beta}^m + \underset{s}{S_l} \varepsilon^{\alpha\beta} X_{l|\beta}^m)]\,. \tag{147}$$

Consequently, after some manipulations, (145) takes the form

$$[y] = \underset{s}{y}\,, \tag{148}$$

where for the spheroidal oscillations

$$\underset{s}{y} = \begin{vmatrix} \underset{s}{U_l} \\ \underset{s}{V_l} \\ 0 \\ 0 \\ \underset{s}{P_l} \\ \underset{s}{R_l} \\ 0 \\ 0 \end{vmatrix}\,, \tag{149}$$

and for toroidal oscillations

$$\underset{s}{y} = \begin{pmatrix} \underset{s}{W_l} \\ \underset{s}{S_l} \end{pmatrix}. \tag{150}$$

Solution of the equations of motion

The adjoint homogeneous boundary value problem: Before going into the solution of the nonhomogeneous boundary value problem, it turns out to be useful first, instead of the present homogeneous boundary value problem, to consider the corresponding adjoint problem.

For this purpose the operator below,

$$\frac{dy}{dr} = Ay\,; \qquad W_0 y_0 + W_a y_a = 0\,, \tag{151}$$

is multiplied from the left by the row vector $\overset{*}{z}$ (complex conjugate of z) and the result is integrated from r_0 to a. In this way

$$\int_{r_0}^{a} \overset{*}{z}\,\frac{dy}{dr}\,dr = \int_{s_0}^{a} \overset{*}{z} A y dr\,.$$

Partial integration of the left-hand side of the above expression gives

$$\Big|_{r_0}^{a} \overset{*}{z} y = \int_{r_0}^{a} \left(\frac{d\overset{*}{z}}{dr} + \overset{*}{z} A\right) y dr\,.$$

Thereafter the use of the boundary conditions of (151) on the left-hand side of the last relation results in the requised adjoint boundary problem

$$\frac{d\overset{*}{z}}{dr} = -\,\overset{*}{z} A\,;\;\; \overset{*}{z}_0 W_0^C + \overset{*}{z}_a W_a^C = 0\,. \tag{152}$$

In (152) the matrices W_0^C and W_a^C are for the spheroidal and toroidal oscillations, respectively,

— Spheroidal oscillations:

$$W_0^C = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad W_a^C = \begin{pmatrix} I & 0 \\ -W & 0 \end{pmatrix}. \tag{153}$$

— Toroidal oscillations:

$$W_0^C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad W_a^C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{154}$$

Formal solution of the nonhomogeneous boundary problem: The nonhomogeneous differential equation (131), together with the homogeneous boundary conditions (139), form the nonhomogeneous boundary value problem

$$\frac{dy}{dr} = Ay + f\,;\;\; W_0 y_0 + W_a y_a = 0\,, \tag{155}$$

where

$$f = -2\varrho_0 p\omega \underset{d}{f} - \underset{s}{f}\,. \tag{156}$$

In *Appendix C* it has been shown that the solution of (155) is

$$y = \int_{r_0}^{a} G(p\,;r\,,s)\; f(s)ds\,, \tag{157}$$

where $G(p\,;r,s)$ is the Green's matrix of (155). The explicit form of $G(p\,;r,s)$ is

$$G(p\,;r\,,s) = \begin{cases} Y(r)D^{-1}W_0\overset{*}{Z}(s)\,; & r \geq s \\ -\,Y(r)D^{-1}W_aY_a\overset{*}{Z}(s)\,; & r \leq s \end{cases}, \tag{158}$$

where $Y(r)$ and $\overset{*}{Z}(s)$ are the fundamental matrices of (151) and (152) respectively and D^{-1} is the inverse of the boundary matrix

$$D(p) = W_0 + W_aY_a\,. \tag{159}$$

As can be seen from (158), the singular points (poles) of the boundary value problem (155) consist of the set $\{p_j\}$, which are solutions of

$$\det D(p) = 0\,. \tag{160}$$

The set $\{p_j\}$ is called the eigenvalue spectrum of the problem.

Because of the importance of $\{p_j\}$ for inverting (157) into the time domain, the following assumptions will be made about the nature of this set:

— $\{p_j\}$ is at most denumerable with no points of condensation in the finite parts of the complex p-plane;

— $\{p_j\}$ is bounded from below, i.e. there is a fixed p_0 such that $|p_j| \geq |p_0|\,,\ \forall p_j \in \{p_j\}$;

— The real part of $\{p_j\}$ is negative, which means that the spectrum lies on the left half of the complex p-plane.

Under these assumptions the inversions of the Laplace transformations in question may be accomplished using the methods of contour integration (on contour integration see BÅTH [6]). Subsequently the solutions (the coefficients with respect to the spherical surface harmonics) are obtained as the residue sum

$$y(r\,,t) = \sum_{n=1}^{\infty} e^{p_n t} \int_{r_0}^{a} Y_n(r) R_n \overset{*}{Z}_n(s) f_n ds\,, \tag{161}$$

where the index n refers to p_n and R_n is the residue contribution

$$R_n = \underset{p_n}{\text{Res}}[R(p)] = \lim_{p \to p_n}(p - p_n)R(p) \tag{162}$$

of

$$R = \begin{cases} D^{-1}W_0 : & r \geq s \\ D^{-1}W_a Y_a ; & r \leq s \end{cases}. \tag{163}$$

As to the focal mechanism (148) (i.e. jump conditions in the displacement and in the normal stress), *Appendix C* shows that the Laplace transformed solution will be

$$y(r\,,p) = G(p\,;r\,,s)\underset{s}{y}\,. \tag{164}$$

The inversion of (164) using methods of contour integration results in the time dependent solution

$$y(r\,,t) = \sum_{n=1}^{\infty} e^{p_n t} Y_n^-(r) R_n \overset{*}{Z}{}_n^-(s) \underset{s}{y}\,. \tag{165}$$

Determination of R_n*:* Seismological literature shows principally two procedures which may be used in evaluating residue contributions.

The direct method involves differentiation of the boundary matrix of the problem partially with respect to p (for this method see Sato *et. al.* [34] and [35]). Since the boundary matrix is usually obtained as a result of a long chain of numerical calculations, it is understandable that numerical differentiation may lead to considerable inaccuracies (i.e. instability).

The second method is based on variational arguments (Rayleigh's principle) and consequently leads to integrations instead of differentiations (on this method see Jeffreys [17] and [18], Takeuchi *et. al.* [39], Keylis-Borok *et. al.* [21], and Harkrider and Anderson [15]. Usually this method gives results more rapidly and more accurately than the first.

Because of the special method of solution used in this paper, it seems appropriate to represent the residue (162) in a form which is particularly suitable for the present purposes.

It is seen in *Appendix C* that the differential identity

$$\frac{dY_n}{dr} = AY_n + (A_n - A)Y_n\,;\; D(p_n) = W_0 + W_a Y_{na}\ \text{(singul.)} \tag{166}$$

may be inverted into the integral equation

$$Y_n(r) = \int_{r_0}^{a} G(p\,;\, r\,,\, s)(A_n(s) - A(s))\,Y_n(s)ds\,,$$

where $G(p; r, s)$ is the Green's matrix (158). With the aid of the explicit expressions (158) and (163) for the Green's matrix the above relation takes the form

$$Y_n(r) = \int_{r_0}^{a} Y(r)R\overset{*}{Z}(s)(A_n(s) - A(s))\,Y_n(s)ds\,. \tag{167}$$

Since the matrix

$$\overset{*}{Z}\,\frac{A_n(r) - A(r)}{p_n - p} \tag{168}$$

is nonsingular, multiplication of (167) from the left by (168) followed by integration from r_0 to a gives

$$\begin{aligned} \int_{r_0}^{a} \overset{*}{Z}(r)\,\frac{A_n(r) - A(r)}{p_n - p}\,Y_n(r)dr &= \int_{r_0}^{a} \overset{*}{Z}(r)\,\frac{A_n(r) - A(r)}{p_n - p}\,Y(r)dr \\ &\int_{r_0}^{a} (p_n - p)[R(p)]\overset{*}{Z}(s)\,\frac{A_n(s) - A(s)}{p_n - p}\,Y_n(s)ds\,. \end{aligned} \tag{169}$$

Since the matrix

$$\int_{r_0}^{a} \overset{*}{Z}(r)\,\frac{A_n(r) - A(r)}{p_n - p}\,Y_n(r)dr \tag{170}$$

is obviously nonsingular, (169) may be multiplied from the right by the inverse of (170) to result in

$$I = \left(\int_{r_0}^{a} \overset{*}{Z}(r)\,\frac{A_n(r) - A(r)}{p_n - p}\,Y(r)dr\right)(p_n - p)[R(p)]\,. \tag{171}$$

Proceeding to the limit in (171) and solving the result for R_n gives the required residue contribution in the form

$$R_n = - \left[\int_{r_0}^{a} \overset{*}{Z}_n(r) \frac{dA_n(r)}{dp} Y_n(r) dr \right]^{-1} . \tag{172}$$

Tidal deformations: The problem concerning the amplitude distribution caused by the tidal forces has been studied extensively in the seismological literature (see *e.g.* the works by TAKEUCHI [37], TAKEUCHI and SAITO [38], ALSOP and KUO [1], etc.). Since the methods of this paper give in compact form the Laplace transformed coefficients (with respect to the surface harmonics) of the dynamic field quantities, it seems appropriate to study the above problem also in this connection.

The problem is stated in mathematical form by the homogeneous part of (131) with the boundary conditions (143). In other words

$$\frac{dy}{dr} = Ay \, ; \; W_0 y_0 + W_a y_a = b \, . \tag{173}$$

With the aid of the fundamental matrix Y the solution of (173) is expressed as (see *Appendix C*)

$$y = Y y_0 \, . \tag{174}$$

For the unknown initial value y_0 in (174), substitution of this relation into the boundary conditions of (173) and solving for y_0 gives

$$y_0 = D^{-1} b \, , \tag{175}$$

where D^{-1} is the inverse of the boundary matrix (159). The required coefficients are now obtained easily from (174) and (175). Thus

$$y = Y D^{-1} b \, . \tag{176}$$

To get the actual forms of the gravitational potential and the displacement vector we ought to solve (176) for Φ_l in the case of the gravitational potential, and solve for U_l, V_l and W_l in the case of the displacement vector. The results are then substituted into (125) and (126), respectively.

Perturbation solution for the rotating sphere: Studies on the free oscillations of a rotating sphere (BACKUS and GILBERT [3], PEKERIS *et. al.* [29]. MacDonald and Ness [24], Gilbert and Backus [12], and Dahlen [9]) have shown that rotation causes splitting of the spectrum (new »lines» appear in the set $\{p_j\}$ and new peaks enter the amplitude spectrum), and in addition the rotation couples the spheroidal and toroidal oscillations.

In this section it is shown that, in the case of a rotating sphere, the set $\{p_j\}$ is influenced by the focal mechanism. Moreover, explicit time dependence (to first order) for the coefficients of the dynamic variables with respect to the surface harmonics will be given.

As seen from (164), the solution corresponding to the focus expressed by the jump relations (145) may be obtained as a special case of the more general solution (161). Therefore it seems appropriate in this section to study only the problem

$$\frac{dy}{dr} = Ay - 2\varrho_0 p\omega \underset{d}{f} - \underset{s}{f}\,;\; W_0 y_0 + W_a y_a = 0\,. \tag{177}$$

According to the usual procedure the solution is sought by a perturbation scheme with respect to the parameter

$$\varepsilon = \frac{\omega}{p^0}\,, \tag{178}$$

where ω is the axial component of the angular velocity of the rotation and p^0 is an eigenvalue of the unperturbed problem. As regards the magnitude of ε it is necessary to assume that

$$|\varepsilon| = \left|\frac{\omega}{p^0}\right| \ll 1\,. \tag{179}$$

The perturbations in p and y are, to first order, of the form

$$p = p^0 + \varepsilon p^1 + \langle\,\rangle \tag{180}$$

and

$$y = y^0 + \varepsilon y^1 + \langle\,\rangle\,. \tag{181}$$

Subsequently, substitution of (180) and (181) into (177) and expansion of A, $\underset{d}{f}$, and $\underset{s}{f}$ in Taylor series around p^0 gives, to first order in ε,

$$\begin{aligned}\frac{dy^0}{dr} + \varepsilon\frac{dy^1}{dr} &= A^0y^0 + \varepsilon\left(A^0y^1 + p^1\frac{dA^0}{dp}\,y^0\right)\\ &\quad - 2\varrho_0 p^{0^2}\varepsilon \underset{d}{f}{}^0 - \underset{s}{f}{}^0 - \varepsilon p^1\frac{d\underset{s}{f}{}^0}{dp} + \langle\,\rangle\,,\end{aligned} \tag{182}$$

where the superscripts zero and one refer to the unperturbed and perturbed states, respectively. (182) is seen to be equivalent to the two simultaneous systems

$$\frac{dy^0}{dr} = A^0 y^0 - f^0 \, ; \, W_0 y_0^0 + W_a y_a^0 = 0 \tag{183}$$

and

$$\frac{dy^1}{dr} = A^0 y^1 + p^1 \frac{dA^0}{dp} y^0 - 2\varrho_0 p^{0^2} \underset{d}{f^0} - p^1 \frac{d\underset{s}{f^0}}{dp} \, ; \; W_0 y_0^1 + W_a y_a^1 = 0 \, . \tag{184}$$

The solution of the unperturbed problem (183) has been obtained in (161). Consequently

$$y^0(r\,,t) = -\sum_{n=1}^{\infty} e^{p_n^0 t} \int_{r_0}^{a} Y_n^0(r) R_n^0 \overset{*}{Z}{}_n^0(s) \underset{s}{f}{}_n^0(s) ds \, . \tag{185}$$

As to (184), it is seen that, before the solution of (184) is obtained, the perturbation contribution $\{p_j^1\}$ to $\{p_j^0\}$ ought to be determined. According to Fredholm's alternative (see e.g. Kantorowitsch and Akilow [20]) the nontrivial solutions of the adjoint boundary value problem ought to be orthogonal to the nonhomogeneous term of (184). In other words

$$p_j^1 \int_{r_0}^{a} \left(\overset{*}{z}{}_j^0 \frac{dA_j^0}{dp} y_j^0 - \overset{*}{z}{}_j^0 \frac{d\underset{s}{f}{}_j^0}{dp} \right) ds = 2\varrho_0 p_j^{0^2} \int_{r_0}^{a} \overset{*}{z}{}_j^0 \underset{d}{f}{}_j^0 ds \, ,$$

from which the required perturbation contribution

$$p_j^1 = 2\varrho p_j^{0^2} \frac{\int_{r_0}^{a} \overset{*}{z}{}_j^0 \underset{d}{f}{}_j^0 ds}{\int_{r_0}^{a} \left(\overset{*}{z}{}_j^0 \frac{dA_j^0}{dp} y_j^0 - \overset{*}{z}{}_j^0 \frac{d\underset{s}{f}{}_j^0}{dp} \right) ds} \tag{186}$$

is easily obtained. In (186) we should notice the interesting fact that, because of the rotation, the set $\{p_j^1\}$ depends on the focal mechanism through $\left\{ \frac{d\underset{d}{f}{}_j^0}{dp} \right\}$.

After solution of $\{p_j^1\}$ the solution $y^1(r, t)$ is obtained in the same way as in the treatment of the unperturbed problem. Accordingly

$$y^1(r\,,t) = \sum_{n=1}^{\infty} e^{p_n^0 t} \int_{r_0}^{a} Y_n^0(r) R_n^0 \overset{*}{Z}{}_n^0(s) f_n^1(s) ds\,, \tag{187}$$

where

$$f_n^1 = p_n^1 \frac{dA_n^0}{dp}\, y_n^0 - 2\varrho_0 p_n^{0^2} f_n^0 - p_n^1 \frac{d\underset{s}{f_n^0}}{dp}\,. \tag{188}$$

p and y are found (to first order) by setting (186) into (180) and (185)—(186) into (181). Thus

$$p_j = p_j^0 \left(1 + 2\varrho_0 \omega \frac{\displaystyle\int_{r_0}^{a} \overset{*}{z}{}_j^0 \underset{d}{f^0} ds}{\displaystyle\int_{r_0}^{a} \left(\overset{*}{z}{}_j^0 \frac{dA_j^0}{dp}\, y_j^0 - \overset{*}{z}{}_j^0 \frac{d\underset{s}{f_j^0}}{dp} \right) ds} \right) \tag{189}$$

and

$$y(r\,,t) = -\sum_{n=1}^{\infty} e^{p_n^0 t} \int_{r_0}^{a} Y_n^0(r) R_n^0 \overset{*}{Z}{}_n^0(s) (\underset{s}{f_n^0}(s) - \varepsilon_n f_n^1(s)) ds\,, \tag{190}$$

where

$$\varepsilon_n = \frac{\omega}{p_n^0}\,.$$

Looking more carefully at the expressions (189) and (190) we see that both involve the term $\underset{d}{f}$. As was observed in (133) and (135), the components of this quantity are $\hat{U}_l$ and $\hat{V}_l$ for the spheroidal oscillations and $\hat{W}_l$ for the toroidal oscillations. However, $\hat{U}_l$, $\hat{V}_l$ and $\hat{W}_l$ were the coefficients of $\boldsymbol{\omega} \times \mathbf{u}$ in the spherical decomposition (129). Therefore, after some work on (129), it may be shown that $\hat{U}_l$, $\hat{V}_l$ and $\hat{W}_l$ are certain linear combinations of U_{l-1}, U_l, U_{l+1}, V_{l-1}, V_l, V_{l+1}, W_{l-1}, W_l, and W_{l+1}. In other words the rotation couples the motions due to the spheroidal and toroidal oscillations (for many aspects of coupling see Dahlen [9]).

Synthetic seismograms: The transformation of (126) into the time domain shows that the displacement vector $\mathbf{u}$ is, in the spherical coordinate system, of the form

$$\mathbf{u}(r, u^1, u^2, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [U_l(r, t)\mathrm{x}^0 X_l^m + \mathbf{a}_\alpha(V_l(r, t)a^{\alpha\beta}X_{l|\beta}^m + W_l(r, t)\varepsilon^{\alpha\beta}X_{l|\beta}^m))] . \tag{191}$$

For a rotating sphere $U_l(r, t)$ and $V_l(r, t)$ are obtained to the first order from (190) in the case of spheroidal oscillations, while $W_l(r, t)$ is obtained from (190) in the case of toroidal oscillations. For a non-rotating sphere $U_l(r, t)$, $V_l(r, t)$ and $W_l(r, t)$ are obtained from (190) in the same way as previously with the exception that now $\varepsilon_n = 0$.

Some aspects of the numerical solution of $Y(\mathrm{r})$ and $\overset{*}{Z}(r)$: Because the solutions of the different boundary value problems were fully based upon the use of the fundamental matrices $Y(r)$ and $\overset{*}{Z}(r)$, it seems necessary to indicate some trends of two numerical processes which may be used in obtaining the numerical values for $Y(r)$ and $\overset{*}{Z}(r)$.

In the first method (see SAASTAMOINEN [32]) y_{i+1} and y_i (the index refers to the respective values of r) are expanded into a Taylor series around $r_{i+\frac{1}{2}}$. Thus, to third order in

$$h_i = r_{i+1} - r_i , \tag{192}$$

we find

$$y_{i+1} = y_{i+\frac{1}{2}} + \frac{h_i}{2} y'_{i+\frac{1}{2}} + \frac{h_i^2}{4} y''_{i+\frac{1}{2}} + \frac{h_i^3}{8} y'''_{i+\frac{\vartheta_1}{2}} \tag{193}$$

and

$$y_i = y_{i+\frac{1}{2}} - \frac{h_i}{2} y'_{i+\frac{1}{2}} + \frac{h_i^2}{4} y''_{i+\frac{1}{2}} - \frac{h_i^3}{8} y'''_{i+\frac{\vartheta_2}{2}} . \tag{194}$$

Since y satisfies the differential equation

$$y' = Ay ,$$

it follows that (193) and (194) are equivalent to the equations

$$y_{i+1} = \left(I + \frac{h_i}{2} A_{i+\frac{1}{2}}\right) y_{i+\frac{1}{2}} + \frac{h_i^3}{8} y'''_{i+\frac{\vartheta_1}{2}} \tag{195}$$

and

$$y_i = \left(I - \frac{h_i}{2} A_{i+\frac{1}{2}}\right) y_{i+\frac{1}{2}} - \frac{h_i^3}{8} y'''_{i+\frac{\vartheta_2}{2}} . \qquad (196)$$

Because $I - \frac{h_i}{2} A_{i+\frac{1}{2}}$ is invertible for sufficiently small h_i, $y_{i+\frac{1}{2}}$ may be eliminated from (195) and (196) to give

$$y_{i+1} = R_i y_i + \langle h_i^3 \rangle , \qquad (197)$$

where

$$R_i = \left(I + \frac{h_i}{2} A_{i+\frac{1}{2}}\right)\left(I - \frac{h_i}{2} A_{i+\frac{1}{2}}\right)^{-1} \qquad (198)$$

and $\langle h_i^3 \rangle$ is a third order matrix quantity under the norm $\| \cdot \|$. In other words

$$\| \langle h_i^3 \rangle \| \leq M h_i^3 . \qquad (199)$$

Starting from fixed y_0 recursive use of (197) gives

$$y(r_n) = R_{n-1} \ldots\ldots R_1 R_0 y_0 + \varepsilon_{n-1} , \qquad (200)$$

where n refers to the number of steps between r_0 and r_n, and

$$\varepsilon_{n-1} = R_{n-1} \ldots\ldots R_1 \langle h_0^3 \rangle + \ldots\ldots + R_{n-1} \langle h_{n-2}^3 \rangle + \langle h_{n-1}^3 \rangle . \qquad (201)$$

In the second method (see Gilbert and Backus [13] and Saastamoinen [31]) the matrix is taken stepwisely constant over a certain number of intervals between r_0 and r_n. In other words

$$A(r) = A_{i+\frac{1}{2}} + \langle \hat{h}_i \rangle ; \; r \in [r_i , r_{i+1}] , \; \forall i = 1 , \ldots , n , \qquad (202)$$

where $\langle \hat{h}_i \rangle$ is a first order matrix quantity. Using (202) the differential equation $y' = Ay$ takes the form

$$y' = (A_{i+\frac{1}{2}} + \langle \hat{h}_i \rangle) y ; \; r \in [r_i , r_{i+1}] .$$

The solution of this equation is of the form

$$y_{i+1} = R_i y_i + \langle h_i \rangle , \qquad (203)$$

where

$$R_i = e^{h_i A_{i+\frac{1}{2}}} \qquad (204)$$

and $\langle h_i \rangle$ is a first order vector quantity under the matrix norm $\| \cdot \|$, or we may say

$$\| \langle h_i \rangle \| \leq M h_i \, . \tag{205}$$

In the same way as before recursive use of (203) gives

$$y(r_n) = R_{n-1} \, \ldots\ldots \, R_1 R_0 \, y_0 + \varepsilon_{n-1} \, , \tag{206}$$

where

$$\varepsilon_{n-1} = R_{n-1} \, \ldots\ldots \, R_1 \langle h_0 \rangle + \ldots\ldots + R_{n-1} \langle h_{n-2} \rangle + \langle h_{n-1} \rangle \, . \tag{207}$$

It is seen from (200) and (206) that a sufficient condition for the convergence of

$$y_n = R_{n-1} \, \ldots\ldots \, R_1 R_0 y_0 \tag{208}$$

nto $y(r_n)$ is

$$\|\varepsilon_{n-1}\| \to 0 \, ; \quad n \to \infty \, . \tag{209}$$

[Remark: In the limiting process (209) it is understood that the end values r_0 and r_n are fixed, while the number of the interval points tends to infinity.]

From (201) and (207) it is seen that the sufficient conditions for (209) to hold are

$$\begin{aligned} &1. \qquad h = \max_{i \leq n} \{h_i\} \to 0 \, ; \quad n \to \infty \, ; \\ &2. \qquad \|R_i\| \leq 1 \, ; \quad \forall i \in \{1 \, , \ldots , n\} \, . \end{aligned} \tag{210}$$

(198) and (204) show that the first condition may be trivially satisfied, while the second condition is satisfied, if the eigenvalues of $A_{i+\frac{1}{2}}$ have negative real parts. Thus a careful inspection of the eigenvalues of the matrix A at different points between r_0 and a may be of great assistance in determining the regions of stability and instability of the actual numerical computations.

It is seen from the above that, under conditions (210), the matrix

$$Y_n = R_{n-1} \, \ldots\ldots \, R_1 R_0 \tag{211}$$

converges with increasing n to the fundamental matrix $Y(r_n)$.

As regards the adjoint fundamental matrix $\overset{*}{Z}(r)$, the relation

$$\overset{*}{Z}(r) Y(r) = I$$

(see *Appendix C*) shows that the regions of stability of $Y(r)$ are regions of instability of $\overset{*}{Z}(r)$.

Summary

As was mentioned earlier, the two main themes in this research work are:

— First to base the derivation of the equations of motion on the axiomatic foundation of modern continuum mechanics.

— Secondly systematic use of matrix notation and especially the use of the fundamental matrices $Y(r)$ and $\overset{*}{Z}(r)$ have made it possible to present the solutions in a concise form.

Concerning the more detailed aspects we may make the following comments:

— The memory models used in the constitutive theory are of the rate type, i. e. the responses depend upon finite numbers of the (material) time derivatives of the Cauchy-Green tensor $\mathbf{C}$, of the temperature ϑ, and of the temperature gradient $\mathbf{h}$ (see *Appendix A*).

— The special expression (172) has been found for the residue contributions at the poles of the integrand.

— As shown in (189) the source mechanism has a small effect on the eigenvalue spectrum $\{p_i\}$ of the rotating sphere.

— Explicit expressions (191) including the first order perturbations have been obtained for the synthetic seismograms of the rotating and nonrotating earth models.

— Two numerical methods for the solution of the fundamental matrices have been studied in some detail. In particular a criterium for studying the stability of the numerical processes has been given.

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REFERENCES

1. ALSOP L. E. and KUO J. T., 1964: The characteristic numbers of semidiurnal earth tidal components for various earth models, *Annales de Géophysique*, **20**, 286–300.
2. ALTERMAN Z., JAROSCH H. and PEKERIS C. L. 1959: Oscillations of the earth, *Proc. Roy. Soc.*, **252**, 80–95.
3. BACKUS G. and GILBERT F. 1961: The rotational splitting of the free oscillations of the earth, *Proc. Natl. Acad. Sci. US*, **47**, 362–371.
4. —»— 1966: Potentials for tangent tensor fields on spheroids, *Arch. Rat. Mech. Anal.*, **22**, 210–252.
5. —»— 1967: Converting vector and tensor equations to scalar equations in spherical coordinates, *Geoph. J. Roy. Astr. Soc.*, **13**, 71–101.
6. BÅTH M. 1968: *Mathematical Aspects of Seismology*, Elsevier Publishing Company.
7. COLE R. H. 1968: *Theory of Ordinary Differential Equations*, Appleton Century Crofts.
8. COLEMAN B. D. 1964: On thermodynamics, strain impulses and viscoelasticity, *Arch. Rat. Mech. Anal.*, **17**, 230–254.
9. DAHLEN, F. A. 1968: The normal modes of a rotating, elliptical earth, *Geoph. J. Roy. Astr. Soc.*, **16**, 329–367.
10. DIEUDONNÉ J. 1960: *Foundations of Modern Analysis*, Academic Press.
11. ERINGEN A. C. 1967: *Mechanics of Continua*, John Wiley & Sons.
12. GILBERT F. and BACKUS G. 1965: Rotational splitting of the free oscillations of the earth 2, *Review of Geophysics* **3**, 1–9.
13. —»— —»— 1966: Propagator matrices in elastic wave and vibration problems, *Geophysics*, **31**, 326–332.
14. GOERTZEL G. and TRALLI N. 1960: *Some Mathematical Methods of Physics*, McGraw-Hill Book Company.
15. HARKRIDER D. G. and ANDERSON D. L. 1966: Surface wave energy from point sources in plane layered earth models, *J. Geoph. Res.*, **71**, 2967–2980.
16. JAUNZEMIS W. 1967: *Continuum Mechanics*, MacMillan Company.
17. JEFFREYS H. 1934: The surface waves from earthquakes, *Month. Not. Roy. Astr. Soc. Geoph.* Suppl., **3**, 253–261.
18. —»— 1961: Small corrections in the theory of surface waves, *Geoph. J. Roy. Astr. Soc.*, **6**, 115–117.

19. JEFFREYS H. 1962: *The Earth*, Cambridge University Press.
20. KANTOROWITSCH K. W. and AKILOW G. P. 1964: *Funktionalanalysis in Normierten Räumen*, Akademie-Verlag, Berlin.
21. KEYLIS-BOROK V. I. and YANOVSKAYA T. B. 1962: Dependence of the spectrum of surface waves on the depth of the focus within earths crust, *Bull. Acad. Sci., USSR Geoph. Ser.*, **11**, 955–959.
22. LOVE A. E. H. 1967: *Some Problems of Geodynamics*, Dover Publications, New York.
23. —»— 1944: *A Treatise on the Mathematical Theory of Elasticity*, Dover Publications, New York.
24. MACDONALD G. J. and NESS N. F. 1961: A study of the free oscillations of the earth, *J. Geoph. Res.*, **66**, 1865–1911.
25. MARTENSEN E. 1968: *Potentialtheorie*, B. G. Teubner, Stuttgart.
26. NOLL W. 1955: On the continuity of the solid and fluid states, *J. Rat. Mech. Anal.*, **4**, 3–81.
27. PACH K. and FREY T. 1964: Vector and tensor analysis, Terra, Budapest.
28. PEKERIS C. L. and JAROSCH H. 1958: The free oscillations of the earth, *Contributions in Geophysics*, **1**, 171–192, Pergamon Press, London.
29. —»— —»— 1961: Rotational multiplets in the spectrum of the earth, *Phys. Rev.*, **122**, 1692–1700.
30. RAYLEIGH Lord 1906: On the dilatational stability of the earth, *Proc. Roy. Soc., London*, A, 77, 486–499.
31. SAASTAMOINEN P. R. 1968: Representation of Rayleigh and Love wave operators in a sphere having rate type viscoelasticity, *Geofysiikan Päivät 18–19. 6. 1968 in Oulu*, (in English).
32. —»— 1968: Oscillations of an isotropic, thermoviscoelastic, radially inhomogeneous sphere, *Geophysica*, **10**, 45–53.
33. SAITO M. 1967: Excitation of free oscillations and surface waves by a point source in a vertically heterogenous earth, *J. Geoph. Res.*, **72**, 3689–3699.
34. SATO Y. and USAMI T. 1964: Propagation of spheroidal disturbances on an elastic sphere with a homogeneous mantle and a core, *Bull. Earthq. Res. Inst.*, **42**, 407–425.
35. —»— —»— and LANDISMAN M 1967: Theoretical seismograms of spheroidal type on the surface of a gravitating elastic sphere, *Ibid.*, **45**, 601–624.
36. SEDOW L. I. 1966: *Foundations of the Nonlinear Mechanics of Continua*, Pergamon Press.
37. TAKEUCHI H. 1950: On the earth tide of the compressible earth of variable density and elasticity, *Trans. Am. Geoph. Union*, **31**, 651–689.
38. —»— and SAITO M 1962: Statical deformations and free oscillations of a model earth, *J. Geoph. Res.*, **67**, 1141–1154.
39. —»— —»— and KOBAYASHI N. 1962: Study of the shear velocity distribution in the upper mantle by Rayleigh and Love waves, *Ibid.*, **67**, 2831–2839.
40. TRUESDELL C. and NOLL W. 1965: The nonlinear field theories of mechanics, *Encyclopedia of Physics III/3*, Springer-Verlag.

Appendix A

Constitutive equations: Because the treatment of the memory models has been restricted in this paper to the rate type thermoviscoelasticity, the responses are supposed to depend on some finite numbers of the (material) time derivatives of $\mathbf{C}$, ϑ and $\mathbf{h}$ (see (1 A) below). In addition all the models studied here ((1 A), (24 A), and (37 A)) satisfy the axioms A, B, C, D, F and G (for further guidance see ERINGEN [11] and TRUESDELL and NOLL [40]).

1. Model:

$$\begin{aligned} \mathbf{T} &= \mathbf{F}\hat{\mathbf{T}}(\mathbf{C}, \dot{\mathbf{C}}, \ldots, \overset{(k}{\mathbf{C}}, \vartheta, \mathbf{h}, \mathbf{X})\mathbf{F}^T \\ \mathbf{q} &= \mathbf{F}\hat{\mathbf{q}}(\mathbf{C}, \dot{\mathbf{C}}, \ldots, \overset{(k)}{\mathbf{C}}, \vartheta, \mathbf{h}, \mathbf{X}) \\ \varepsilon &= \hat{\varepsilon}(\mathbf{C}, \dot{\mathbf{C}}, \ldots, \overset{(k)}{\mathbf{C}}, \vartheta, \mathbf{h}, \mathbf{X}) \\ \eta &= \hat{\eta}(\mathbf{C}, \dot{\mathbf{C}}, \ldots, \overset{(k)}{\mathbf{C}}, \vartheta, \mathbf{h}, \mathbf{X}) \\ \psi &= \hat{\psi}(\mathbf{C}, \mathbf{C}, \ldots, \overset{(k)}{\mathbf{C}}, \vartheta, \mathbf{h}, \mathbf{X}) \end{aligned} \tag{1 A}$$

(1 A) shows that the material responses map the product space

$$H_k = \prod_k L(V_t\,; V_t) \times \mathfrak{R}_t \times V_t \tag{2 A}$$

nto $L(V_t; V_t)$, V_t, $\mathfrak{R}_t$, $\mathfrak{R}_t$ and $\mathfrak{R}_t$ respectively. In (2 A) $\prod_k L(V_t\,; V_t)$ $= \overbrace{L(V_t\,; V_t) \times \ldots \times L(V_t\,; V_t)}^{k}$. The norm in H_k will be the sum of the norms of the component spaces.

According to axiom H the responses (1 A) ought to satisfy the Clausius-Duhem inequality (108). The time derivative of ψ needed in this connection will be obtained according to (18). Thus

$$\dot{\psi} = \partial_{\mathbf{C}}\hat{\psi}[\dot{\mathbf{C}}] + \sum_{i=1}^{k} \partial_{\mathbf{C}}^{(i)}\hat{\psi}[\overset{(i+1)}{\mathbf{C}}] + \partial_{\vartheta}\hat{\psi}\dot{\vartheta} + \partial_{\mathbf{h}}\hat{\psi}[\dot{\mathbf{h}}]\,, \tag{3 A}$$

where $\partial_{\mathbf{C}}\hat{\psi}[\;]$ and $\partial^{(i)}[\;] \in L(L(V_t\,; V_t)\,; \mathfrak{R}_t)$, $\partial_{\vartheta}\hat{\psi} \in L(\mathfrak{R}_t\,; \mathfrak{R}_t)$ and $\partial_{\mathbf{h}}\hat{\psi}[\;] \in L(V_t\,; \mathfrak{R}_t)$. Substitution of (3 A) into the C-D inequality then gives

$$\tfrac{1}{2}\,\mathrm{tr}[(\mathbf{T}^* - 2\,\partial_{\smile}\hat{\psi})\dot{\mathbf{C}}] - \sum_{i=1}^{k} \partial^{(i)}_{\smile}\hat{\psi}[\overset{(i+1)}{\mathbf{C}}] - \dot{\vartheta}(\eta + \partial_{\vartheta}\hat{\psi}) - \partial_{\mathrm{h}}\hat{\psi}[\dot{\mathbf{h}}] + \frac{\mathbf{h}\cdot\mathbf{q}^*}{\vartheta} \geq 0\,. \tag{4 A}$$

Since $\overset{(k+1)}{\mathbf{C}}$, $\dot{\vartheta}$ and $\dot{\mathbf{h}} \notin H_k$, it follows from (4 A) that

$$\partial^{(k)}_{\smile}\hat{\psi} = 0\,, \qquad \partial_{\mathrm{h}}\hat{\psi} = 0 \tag{5 A}$$

and

$$\eta = -\,\partial_{\vartheta}\hat{\psi}\,. \tag{6 A}$$

$\mathbf{T}^*$ may be decomposed into the sum of an elastic part and a dissipative part as follows:

$$\mathbf{T}^* = \mathbf{T}^*_E + \mathbf{T}^*_D\,, \tag{7 A}$$

where the elastic part $\mathbf{T}^*_E$ is defined by

$$\mathbf{T}^*_E = 2\,\partial_{\smile}\hat{\psi}\,. \tag{8 A}$$

The dissipative part $\mathbf{T}^*_D$ is found to satisfy the inequality

$$\tfrac{1}{2}\,\mathrm{tr}(\mathbf{T}^*_D\dot{\mathbf{C}}) - \sum_{i=1}^{k-1} \partial^{(i)}_{\mathbf{C}}\hat{\psi}[\overset{(i+1)}{\mathbf{C}}] + \frac{\mathbf{h}\cdot\mathbf{q}^*}{\vartheta} \geq 0\,. \tag{9 A}$$

Further restrictions (besides (5 A)) are obtained from (9 A) after $\mathbf{T}^*_D$, ψ and $\mathbf{q}^*$ have been expanded in Taylor series around the element

$$\underset{*}{H_k} = (\mathbf{C}\,, 0\,, \ldots\,, 0\,, \vartheta\,, 0) \in H_k\,. \tag{10 A}$$

Consequently

$$\underset{*}{\mathbf{T}^*_D} = 0\,,\ \ \underset{*}{\partial^{(i)}_{\smile}\hat{\psi}} = 0 \ \ (i = 1\,, \ldots\,, k-1) \ \text{ and } \ \underset{*}{\mathbf{q}^*} = 0\,. \tag{11 A}$$

Finally, Taylor expansions of ψ , η (from (6 A)) , $\mathbf{T}^*$ (from (7 A)), and $\mathbf{q}^*$ around the element

$$\underset{0}{H_k} = (I\,, 0\,, \ldots\,, 0\,, \vartheta_0\,, 0) \in H_k \tag{12 A}$$

give for the infinitesimal constitutive relations the expressions

$$\psi = \underset{0}{\hat{\psi}} + \partial_{\mathrm{E}}\underset{0}{\hat{\psi}}[\mathbf{E}] + \partial_{\vartheta}\underset{0}{\hat{\psi}}T + \tfrac{1}{2}\,\partial_{\mathrm{EE}}\underset{0}{\hat{\psi}}[\mathbf{E}\,, \mathbf{E}] + \partial_{\mathrm{E}\vartheta}\underset{0}{\psi}[\mathbf{E}]T + \tfrac{1}{2}\,\partial_{\vartheta\vartheta}\hat{\psi}T^2 + \langle\,\rangle\,, \tag{13 A}$$

$$\eta = -\partial_{\vartheta}\underset{0}{\hat{\psi}} - \partial_{\mathbf{E}\vartheta}\underset{0}{\hat{\psi}}[\mathbf{E}] - \partial_{\vartheta\vartheta}\underset{0}{\hat{\psi}}T + \langle\,\rangle\,, \tag{14 A}$$

$$\mathbf{T}^* = \partial_{\mathbf{E}}\underset{0}{\hat{\psi}} + \partial_{\mathbf{EE}}\underset{0}{\hat{\psi}}[\mathbf{E}] + \partial_{\mathbf{E}\vartheta}\underset{0}{\hat{\psi}}T + \sum_{i=1}^{k}\partial_{\mathbf{E}}^{(i)}\underset{0}{\mathbf{T}_D}[\overset{(i)}{\mathbf{E}}] + \partial_{\mathbf{h}}\underset{0}{\mathbf{T}_D}[\mathbf{h}] + \langle\,\rangle\,, \tag{15 A}$$

and

$$\mathbf{q}^* = \sum_{i=1}^{k}\partial_{\mathbf{E}}^{(i)}\underset{0}{\mathbf{q}}[\overset{(i)}{\mathbf{E}}] + \partial_{\mathbf{h}}\underset{0}{\mathbf{q}}[\mathbf{h}] + \langle\,\rangle\,. \tag{16 A}$$

It is found that in (13 A)—(16 A) $\mathbf{C}$ has been replaced by the corresponding strain tensor $\mathbf{E}$ (see (45)) and ϑ has been decomposed into

$$\vartheta = \vartheta_0 + T\,, \tag{17 A}$$

where ϑ_0 is the static part and T is the infinitesimal dynamic part of ϑ.

Thereafter, through use of axiom E, the relations (52)—(53), and the Laplace transformation (54), the following relations are obtained for the (Laplace transformed and spatial) stress tensor $\mathbf{T}$ and the heat conduction vector $\mathbf{q}$ in isotropic and transversely isotropic symmetries.

— Isotropic symmetry:

$$\mathbf{T} = \frac{p_0\mathbf{I}}{p} - \mathbf{u}\cdot\partial_{\mathbf{x}}p_0\mathbf{I} - \gamma T\mathbf{I} + \tilde{\lambda}\,\mathrm{tr}\,\mathbf{e}\mathbf{I} + 2\tilde{\mu}\mathbf{e} + \langle\,\rangle \tag{18 A}$$

and

$$\mathbf{q} = \varkappa\partial_{\mathbf{x}}T + \langle\,\rangle\,. \tag{19A}$$

In (18 A) and (19 A) $p_0\mathbf{I}$ is the hydrostatic tension, γ is the stress temperature modulus, $\varkappa$ is the heat conduction modulus, and $\tilde{\lambda}$ and $\tilde{\mu}$ are the Laplace transformed stress-strain moduli given by

$$\tilde{\lambda} = \lambda - p_0 + \sum_{i=1}^{k}\lambda_i p^i \quad\text{and}\quad \tilde{\mu} = \mu + p_0 + \sum_{i=1}^{k}\mu_i p^i\,. \tag{20 A}$$

— Transversely isotropic symmetry:

$$\begin{aligned}\mathbf{T} = {} & \frac{p_0\mathbf{I}}{p} - \mathbf{u}\cdot\partial\,p_0\mathbf{I} - (\gamma\mathbf{I} + \mathbf{x}^0\otimes\mathbf{x}^0\gamma')T + \mathbf{x}^0\otimes\mathbf{x}^0(\tilde{\beta}e_{rr} + \tilde{\lambda}e_\gamma^\gamma) \\ & + (\mathbf{x}^0\otimes\mathbf{a}_\alpha + \mathbf{a}_\alpha\otimes\mathbf{x}^0)2\tilde{\mu}e^{r\alpha} + \mathbf{a}_\alpha\otimes\mathbf{a}_\beta\,[a^{\alpha\beta}(\tilde{\lambda}'e_\gamma^\gamma + \tilde{\lambda}e_{rr}) \\ & + 2\tilde{\mu}'e^{\alpha\beta}] + \langle\,\rangle\end{aligned} \tag{21 A}$$

and

$$\mathbf{q} = \tilde{\varkappa}\partial_{\mathbf{x}}T + \tilde{\varkappa}'T_{,r}\mathbf{x}^0 + \langle\,\rangle\,, \tag{22 A}$$

where γ and γ' are the stress temperature moduli, $\varkappa$ and $\varkappa'$ are the heat conduction moduli, and

$$\begin{aligned}&\tilde{\beta} = \beta + p_0 + \sum_{i=1}^{k}\beta_i p^i\,,\ \tilde{\lambda} = \lambda - p_0 + \sum_{i=1}^{k}\lambda_i p^i\,,\ \tilde{\lambda}' = \lambda' - p_0 + \sum_{i=1}^{k}\lambda_i' p^i\\ &\tilde{\mu} = \mu + p_0 + \sum_{i=1}^{k}\mu_i p^i \ \text{ and } \ \tilde{\mu}' = \mu' + p_0 + \sum_{i=1}^{k}\mu_i' p^i\end{aligned} \tag{23 A}$$

are the Laplace transformed stress-strain moduli.

2. Memory model:

In this case the model is characterized by the responses

$$\begin{aligned}\mathbf{T} &= \mathbf{F}\hat{\mathbf{T}}(\mathbf{C}\,,\dot{\mathbf{C}}\,,\ldots,\overset{(k)}{\mathbf{C}}\,,\vartheta\,,\dot{\vartheta}\,,\ldots,\overset{(l)}{\vartheta}\,,\mathbf{h}\,,\mathbf{X})\mathbf{F}^T\\ \mathbf{q} &= \mathbf{F}\hat{\mathbf{q}}(\mathbf{C}\,,\dot{\mathbf{C}}\,,\ldots,\overset{(k)}{\mathbf{C}}\,,\vartheta\,,\dot{\vartheta}\,,\ldots,\overset{(l)}{\vartheta}\,,\mathbf{h}\,,\mathbf{X})\\ \varepsilon &= \hat{\varepsilon}(\mathbf{C}\,,\dot{\mathbf{C}}\,,\ldots,\overset{(k)}{\mathbf{C}}\,,\vartheta\,,\dot{\vartheta}\,,\ldots,\overset{(l)}{\vartheta}\,,\mathbf{h}\,,\mathbf{X})\\ \eta &= \hat{\eta}(\mathbf{C}\,,\dot{\mathbf{C}}\,,\ldots,\overset{(k)}{\mathbf{C}}\,,\vartheta\,,\dot{\vartheta}\,,\ldots,\overset{(l)}{\vartheta}\,,\mathbf{h}\,,\mathbf{X})\\ \psi &= \hat{\psi}(\mathbf{C}\,,\dot{\mathbf{C}}\,,\ldots,\overset{(k)}{\mathbf{C}}\,,\vartheta\,,\dot{\vartheta}\,,\ldots,\overset{(l)}{\vartheta}\,,\mathbf{h}\,,\mathbf{X})\,.\end{aligned} \tag{24 A}$$

Because the responses have been defined on the product space

$$H_{kl} = \prod_k (L(V_t\,;V_t) \times \prod_l \mathfrak{R}_t \times V_t\,, \tag{25 A}$$

only a few modifications to the previous discussion on the first model are needed to show that the infinitesimal responses $\psi\,,\eta\,,$ $\mathbf{T}^*$, and $\mathbf{q}^*$ around the element (equilibrium state)

$$\underset{0}{H_{kl}} = (\mathbf{I}\,,0\,,\ldots,0\,,\vartheta_0\,,0\,,\ldots,0\,,0) \in H_{kl} \tag{26 A}$$

are of the form

$$\begin{aligned}\psi &= \underset{0}{\hat{\psi}} + \partial_{\cdot}\underset{0}{\hat{\psi}}[\mathbf{E}] + \partial_{\vartheta}\underset{0}{\hat{\psi}}T + \tfrac{1}{2}\,\partial_{EE}\underset{0}{\hat{\psi}}[\mathbf{E}\,,\mathbf{E}] + \partial_{E\vartheta}\underset{0}{\hat{\psi}}[\mathbf{E}]T\\ &\quad + \tfrac{1}{2}\,\partial_{\vartheta\vartheta}\underset{0}{\hat{\psi}}T^2 + \langle\,\rangle\,,\end{aligned} \tag{27 A}$$

$$\eta = -\partial_{\vartheta}\hat{\underset{0}{\psi}} - \partial_{E\vartheta}\hat{\underset{0}{\psi}}[\mathbf{E}] - \partial_{\vartheta\vartheta}\hat{\underset{0}{\psi}}T - \sum_{i=1}^{k} \partial_{E}^{(i)}\overset{(i)}{\underset{0}{\eta_D}}[\mathbf{E}] - \sum_{i=1}^{l} \partial_{\vartheta}^{(i)}\underset{0}{\eta_D}\overset{(i)}{T} - \partial_{\mathbf{h}}\underset{0}{\eta_D}[\mathbf{h}] + \langle\,\rangle, \tag{28 A}$$

$$\mathbf{T}^* = \partial_{E}\hat{\underset{0}{\psi}} + \partial_{EE}\hat{\underset{0}{\psi}}[\mathbf{E}] + \partial_{E\vartheta}\hat{\underset{0}{\psi}}T + \sum_{i=1}^{k} \partial_{E}^{(i)}\underset{0}{\mathbf{T}_D}\overset{(i)}{[\mathbf{E}]} + \sum_{i=1}^{l} \partial_{E}^{(i)}\underset{0}{\mathbf{T}_D}\overset{(i)}{T} + \partial_{h}\underset{0}{\mathbf{T}_D}[\mathbf{h}] + \langle\,\rangle, \tag{29 A}$$

and

$$\mathbf{q}^* = \sum_{i=1}^{k} \partial_{E}^{(i)}\underset{0}{\mathbf{q}}\overset{(i)}{[\mathbf{E}]} + \sum_{i=1}^{l} \partial_{\vartheta}^{(i)}\underset{0}{\mathbf{q}}\overset{(i)}{T} + \partial_{\mathbf{h}}\underset{0}{\mathbf{q}}[\mathbf{h}] + \langle\,\rangle. \tag{30 A}$$

In the same way as previously (in model 1.) it may be shown that the Laplace transformed forms of $\mathbf{T}$ and $\mathbf{q}$ are, in the spatial coordinate system and in the two symmetry classes, respectively:

— Isotropic symmetry:

$$\mathbf{T} = \frac{p_0\mathbf{I}}{p} - \mathbf{u}\cdot\partial_{\mathbf{x}}p_0\mathbf{I} - \tilde{\gamma}T\mathbf{I} + \tilde{\lambda}\operatorname{tr}\mathbf{e}\mathbf{I} + 2\tilde{\mu}\mathbf{e} + \langle\,\rangle \tag{31 A}$$

and

$$\mathbf{q} = \varkappa\partial_{\mathbf{x}}T + \langle\,\rangle, \tag{32 A}$$

where the other moduli are the same as in (18 A)—(19 A), except that

$$\tilde{\gamma} = \gamma + \sum_{i=1}^{l} \gamma_i p^i. \tag{33 A}$$

— Transversely isotropic symmetry:

$$\mathbf{T} = \frac{p_0\mathbf{I}}{p} - \mathbf{u}\cdot\partial_{\mathbf{x}}p\mathbf{I} - (\tilde{\gamma}\mathbf{I} + \mathbf{x}^0\otimes\mathbf{x}^0\tilde{\gamma}')T + \mathbf{x}^0\otimes\mathbf{x}^0(\tilde{\beta}e_{rr} + \tilde{\lambda}e^{\gamma}_{\gamma}) + (\mathbf{x}^0\otimes\mathbf{a}_\alpha + \mathbf{a}_\alpha\otimes\mathbf{x}^0)2\tilde{\mu}e^{r\alpha} + \mathbf{a}_\alpha\otimes\mathbf{a}_\beta[a^{\alpha\beta}(\tilde{\lambda}'e^{\gamma}_{\gamma} + \tilde{\lambda}e_{rr}) + 2\tilde{\mu}'e^{\alpha\beta}] + \langle\,\rangle \tag{34 A}$$

and

$$\mathbf{q} = (\varkappa\mathbf{I} + \mathbf{x}^0\otimes\mathbf{x}^0\varkappa')\partial_{\mathbf{x}}T + \langle\,\rangle, \tag{35 A}$$

where the other moduli are the same as in (21 A)—(22 A), except that

$$\tilde{\gamma} = \gamma + \sum_{i=1}^{l} \gamma_i p^i \quad \text{and} \quad \tilde{\gamma}' = \gamma' + \sum_{i=1}^{l} \gamma'_i p^i \,. \tag{36 A}$$

3. Memory model:

In this model the responses are of the form

$$\begin{aligned}
\mathbf{T} &= \mathbf{F}\hat{\mathbf{T}}(\mathbf{C}, \dot{\mathbf{C}}, \ldots, \overset{(k)}{\mathbf{C}}, \vartheta, \dot{\vartheta}, \ldots, \overset{(l)}{\vartheta}, \mathbf{h}, \dot{\mathbf{h}}, \ldots, \overset{(m)}{\mathbf{h}}, \mathbf{X})\mathbf{F}^T \\
\mathbf{q} &= \hat{\mathbf{q}}(\mathbf{C}, \dot{\mathbf{C}}, \ldots, \overset{(k)}{\mathbf{C}}, \vartheta, \dot{\vartheta}, \ldots, \overset{(l)}{\vartheta} . \mathbf{h}, \dot{\mathbf{h}}, \ldots, \overset{(m)}{\mathbf{h}}, \mathbf{X}) \\
\varepsilon &= \hat{\varepsilon}(\mathbf{C}, \dot{\mathbf{C}}, \ldots, \overset{(k)}{\mathbf{C}}, \vartheta, \dot{\vartheta}, \ldots, \overset{(l)}{\vartheta}, \mathbf{h}, \dot{\mathbf{h}}, \ldots, \overset{(m)}{\mathbf{h}}, \mathbf{X}) \\
\eta &= \hat{\eta}(\mathbf{C}, \dot{\mathbf{C}}, \ldots, \overset{(k)}{\mathbf{C}}, \vartheta, \dot{\vartheta}, \ldots, \overset{(l)}{\vartheta}, \mathbf{h}, \dot{\mathbf{h}}, \ldots, \overset{(m)}{\mathbf{h}}, \mathbf{X}) \\
\psi &= \hat{\psi}(\mathbf{C}, \dot{\mathbf{C}}, \ldots, \overset{(k)}{\mathbf{C}}, \vartheta, \dot{\vartheta}, \ldots, \overset{(l)}{\vartheta}, \mathbf{h}, \dot{\mathbf{h}}, \ldots, \overset{(m)}{\mathbf{h}}, \mathbf{X})
\end{aligned} \tag{37 A}$$

and are defined on the product space

$$H_{klm} = \prod_k L(V_\iota\,; V_\iota) \times \prod_l \mathfrak{R}_\iota \times \prod_m V_\iota \,. \tag{38 A}$$

The infinitesimal responses around the element

$$\underset{0}{H}_{klm} = (\mathbf{I}, 0, \ldots, 0, \vartheta_0, 0, \ldots, 0, 0) \in H_{klm} \tag{39 A}$$

are obtained basically in the same way as before. Thus,

$$\begin{aligned}
\psi &= \underset{0}{\hat{\psi}} + \partial_{\mathrm{E}}\underset{0}{\hat{\psi}}[\mathbf{E}] + \partial_\vartheta \underset{0}{\hat{\psi}} T + \tfrac{1}{2}\,\partial_{\mathrm{EE}}\underset{0}{\hat{\psi}}[\mathbf{E}, \mathbf{E}] + \partial_{\mathrm{E}\vartheta}\underset{0}{\hat{\psi}}[\mathbf{E}]T\,, \\
&\quad + \tfrac{1}{2}\,\partial_{\vartheta\vartheta}\underset{0}{\hat{\psi}} T^2 + \langle\,\rangle\,,
\end{aligned} \tag{40 A}$$

$$\begin{aligned}
\eta &= -\,\partial_\vartheta \underset{0}{\hat{\psi}} - \partial_{\mathrm{E}\vartheta}\underset{0}{\hat{\psi}}[\mathbf{E}] - \partial_{\vartheta\vartheta}\underset{0}{\hat{\psi}} T - \sum_{i=1}^{k} \partial_{\mathrm{E}}^{(i)} \underset{0}{\eta_D}[\overset{(i)}{\mathbf{E}}] - \sum_{i=1}^{l} \partial_{\vartheta}^{(i)} \underset{0}{\eta_D} \overset{(i)}{T} \\
&\quad - \sum_{i=0}^{m} \partial_{\mathrm{h}}^{(i)} \underset{0}{\eta_D}[\overset{(i)}{\mathbf{h}}] + \langle\,\rangle\,,
\end{aligned} \tag{41 A}$$

$$\begin{aligned}
\mathbf{T}^* &= \partial_{\mathrm{E}}\underset{0}{\hat{\psi}} + \partial_{\iota\,\mathrm{E}}\underset{0}{\hat{\psi}}[\mathbf{E}] + \partial_{\mathrm{E}\vartheta}\underset{0}{\hat{\psi}} T + \sum_{i=1}^{k} \partial_{\mathrm{E}}^{(i)} \underset{0}{\mathbf{T}_D}[\overset{(i)}{\mathbf{E}}] + \sum_{i=1}^{l} \partial_{\vartheta}^{(i)} \underset{0}{\mathbf{T}_D} \overset{(i)}{T} \\
&\quad + \sum_{i=0}^{m} \partial_{\mathrm{h}}^{(i)} \underset{0}{\mathbf{T}_D}[\overset{(i)}{\mathbf{h}}] + \langle\,\rangle\,,
\end{aligned} \tag{42 A}$$

and

$$\mathbf{q}^* = \sum_{i=1}^{k} \partial_{\mathbf{E}}^{(i)} \mathbf{q}^* [\overset{(i)}{\underset{0}{\mathbf{E}}}] + \sum_{i=1}^{l} \partial_{\vartheta}^{(i)} \mathbf{q}^* \overset{(i)}{\underset{0}{T}} + \sum_{i=0}^{m} \partial_{\mathbf{h}}^{(i)} \mathbf{q}^* [\overset{(i)}{\underset{0}{\mathbf{h}}}] + \langle\,\rangle \,. \tag{43 A}$$

The Laplace transformed forms of (42 A)—(43 A) are, in the spatial coordinate system and in the two symmetry classes, of the form —

— Isotropic symmetry:

$$\mathbf{T} = \frac{p_0 \mathbf{I}}{p} - \mathbf{u} \cdot \partial_{\mathbf{x}} p_0 \mathbf{I} - \hat{\gamma} T \mathbf{I} + \tilde{\lambda} \operatorname{tr} \mathbf{e} \mathbf{I} + 2 \tilde{\mu} \mathbf{e} + \langle\,\rangle \tag{44 A}$$

and

$$\mathbf{q} = \tilde{\varkappa} \partial_{\mathbf{x}} T + \langle\,\rangle \,, \tag{45 A}$$

where the other moduli are the same as in (31 A)—(32 A), except that

$$\tilde{\varkappa} = \varkappa + \sum_{i=1}^{m} \varkappa_i p^i \,. \tag{46 A}$$

— Transversely isotropic symmetry:

$$\begin{aligned} \mathbf{T} = {} & \frac{p_0 \mathbf{I}}{p} - \mathbf{u} \cdot \partial_{\mathbf{x}} p_0 \mathbf{I} - (\tilde{\gamma} \mathbf{I} + \mathbf{x}^0 \otimes \mathbf{x}^0 \tilde{\gamma}') T + \mathbf{x}^0 \otimes \mathbf{x}^0 (\tilde{\beta} e_{rr} + \tilde{\lambda} e_{\gamma}^{\gamma}) \\ & + (\mathbf{x}^0 \otimes \mathbf{a}_\alpha + \mathbf{a}_\alpha \otimes \mathbf{x}^0)\, 2 \tilde{\mu} e^{r\alpha} + \mathbf{a}_\alpha \otimes \mathbf{a}_\beta \,[a^{\alpha\beta} (\tilde{\lambda}' e_{\gamma}^{\gamma} + \tilde{\lambda} e_{rr}) \\ & + 2 \tilde{\mu}' e^{\alpha\beta}] + \langle\,\rangle \end{aligned} \tag{47 A}$$

and

$$\mathbf{q} = (\tilde{\varkappa} \mathbf{I} + \mathbf{x}^0 \otimes \mathbf{x}^0 \tilde{\varkappa}') \partial_{\mathbf{x}} T + \langle\,\rangle \,, \tag{48 A}$$

where the other moduli are the same as in (34 A)—(35 A), except that now

$$\tilde{\varkappa} = \varkappa + \sum_{i=1}^{m} \varkappa_i p^i \quad \text{and} \quad \tilde{\varkappa}' = \varkappa' + \sum_{i=1}^{m} \varkappa_i' p^i \,. \tag{49 A}$$

Appendix B

Energy equation for infinitesimal deformations: After h has been set as zero in the energy equation (99), it follows

$$\varrho \dot{\varepsilon} = \frac{\varrho}{2} \operatorname{tr} (\mathbf{T}^* \dot{\mathbf{C}}) + \operatorname{div} \mathbf{q} \,. \tag{1 B}$$

Since ε has been defined by

$$\varepsilon = \psi + \vartheta\eta ,$$

the (material) time derivative of ε will be

$$\dot{\varepsilon} = \dot{\psi} + \dot{T}\eta + \vartheta\dot{\eta} , \tag{2 B}$$

where the relation (17 A) has been used for ϑ

Differentiating the infinitesimal relations for ψ obtained in *Appendix A* with respect to time it may be shown that in all the memory models

$$\dot{\psi} = \underset{0}{\partial_{\mathrm{E}}\hat{\psi}}[\dot{\mathbf{E}}] + \underset{0}{\partial_{\vartheta}\hat{\psi}}\dot{T} + \langle\,\rangle . \tag{3 B}$$

In the same way, using the results of *Appendix A*, it may be shown that

$$\dot{T}\eta = -\underset{0}{\partial_{\vartheta}\hat{\psi}}\dot{T} + \langle\,\rangle . \tag{4 B}$$

The last term on the right-hand side of (2 B) has the three different expressions given below, in accordance with the memory model used.

In model 1:

$$\vartheta\dot{\eta} = -\vartheta_0 \underset{0}{\partial_{\mathrm{E}\vartheta}\hat{\psi}}[\dot{\mathbf{E}}] - \vartheta_0 \underset{0}{\partial_{\vartheta\vartheta}\hat{\psi}}\dot{T} + \langle\,\rangle ; \tag{5 B}$$

In model 2:

$$\begin{aligned} \vartheta\dot{\eta} = & -\vartheta_0 \underset{0}{\partial_{\mathrm{E}\vartheta}\hat{\psi}}[\dot{\mathbf{E}}] - \vartheta_0 \underset{0}{\partial_{\vartheta\vartheta}\hat{\psi}}\dot{T} - \vartheta_0 \sum_{i=1}^{k} \underset{0}{\partial_{\mathrm{E}}^{(i)}\eta_D}[\overset{(i+1)}{\mathbf{E}}] - \vartheta_0 \sum_{i=1}^{l} \underset{0}{\partial_{\vartheta}^{(i)}\eta_D}\overset{(i+1)}{T} \\ & - \vartheta_0 \underset{0}{\partial_{\mathrm{h}}\eta_D}[\dot{\mathrm{h}}] + \langle\,\rangle ; \end{aligned} \tag{6 B}$$

In model 3:

$$\begin{aligned} \vartheta\dot{\eta} = & -\vartheta_0 \underset{0}{\partial_{\mathrm{E}\vartheta}\hat{\psi}}[\dot{\mathbf{E}}] - \vartheta_0 \underset{0}{\partial_{\vartheta\vartheta}\hat{\psi}}\dot{T} - \vartheta_0 \sum_{i=1}^{k} \underset{0}{\partial_{\mathrm{E}}^{(i)}\eta_D}[\overset{(i+1)}{\mathbf{E}}] - \vartheta_0 \sum_{i=1}^{l} \underset{0}{\partial_{\vartheta}^{(i)}\eta_D}\overset{(i+1)}{T} \\ & - \vartheta_0 \sum_{i=0}^{m} \underset{0}{\partial_{\mathrm{h}}\eta_D}[\overset{(i+1)}{\mathbf{h}}] + \langle\,\rangle . \end{aligned} \tag{7 B}$$

Hence, instead of ε, the relation (2 B) is substituted together with (3 B), (4 B), (5 B), (6 B), and (7 B) into the left-hand of (1 B), while the infinitesimal relations for $\mathbf{T}^*$ from *Appendix A*, together with the relation obtained differentiating (45) with respect to time, are substituted into the right-hand side of (1 B). In this way it may be shown that the Laplace transformed form of the energy equation (1 B), in the spatial coordinate system and in the two symmetry classes, takes on the following forms:

— Isotropic symmetry:

$$\operatorname{div} \mathbf{q} = \varrho_0 p \tilde{c}_E T + \vartheta_0 p \underset{0}{\tilde{\gamma}} \operatorname{div} \mathbf{u} + \langle\,\rangle\,, \tag{8 B}$$

where for the memory model 1:

$$\tilde{c}_E = c_E \quad \text{and} \quad \underset{0}{\tilde{\gamma}} = \gamma \tag{9 B}$$

and for the memory models 2 and 3:

$$\tilde{c}_E = c_E + \sum_{i=1}^{l} c_i p^i \quad \text{and} \quad \underset{0}{\tilde{\gamma}} = \gamma + \sum_{i=1}^{k} \hat{\gamma}_i p^i\,. \tag{10 B}$$

— Transversely isotropic symmetry:

$$\operatorname{div} \mathbf{q} = \varrho_0 p \tilde{c}_E T + \vartheta_0 p \underset{0}{\tilde{\gamma}} \operatorname{div} \mathbf{u} + \vartheta_0 p \underset{0}{\tilde{\gamma}}' u_{r,r} + \langle\,\rangle\,, \tag{11 B}$$

where for the memory model 1:

$$\tilde{c}_E = c_E\,, \quad \underset{0}{\tilde{\gamma}} = \gamma \quad \text{and} \quad \underset{0}{\tilde{\gamma}}' = \gamma' \tag{12 B}$$

and for the memory models 2 and 3:

$$\tilde{c}_E = c_E + \sum_{i=1}^{l} c_i p^i\,, \quad \underset{0}{\tilde{\gamma}} = \gamma + \sum_{i=1}^{k} \hat{\gamma}_i p^i \quad \text{and} \quad \underset{0}{\tilde{\gamma}}' = \gamma' + \sum_{i=1}^{k} \hat{\gamma}'_i p^i \tag{13 B}$$

Appendix C

Formal solutions of nonhomogeneous vector differential equation with homogeneous boundary conditions: In mathematical form the problem is to solve

$$\frac{dy}{dr} = Ay + f\,; \quad W_0 y_0 + W_a y_a = 0\,, \tag{1 C}$$

where y, f, y_0, and y_a are certain column vectors and A, W_0, and W_a are square matrices.

Since the solution of the homogeneous initial value problem

$$\frac{dy}{dr} = Ay\,; \quad y(r_0) = y_0 \tag{2 C}$$

is needed in what follows, this is studied first in some detail. Under the assumption of uniqueness, the solution of (2 C) may be regarded as a linear mapping $Y(r\,, r_0)$ from the set of initial vectors $\{y_0\}$ into the set of solutions $\{y(r)\}$. In other words

$$y(r) = Y(r\,, r_0)y_0\,. \tag{3 C}$$

The initial condition of (2 C) demands that

$$Y(r_0\,, r_0) = I\,. \tag{4 C}$$

Because (3 C) is valid for all $y_0 \in \{y_0\}$, it follows from (2 C) and (4 C) that the fundamental matrix $Y(r\,, r_0)$ satisfies the initial value problem

$$\frac{dY}{dr} = AY\,; \quad Y(r_0\,, r_0) = I\,. \tag{5 C}$$

As to the Green's function of (1 C), the treatment here follows Cole [7]. Accordingly the initial value problem

$$\frac{dy}{dr} = Ay + f\,; \quad y(r_0) = y_0 \tag{6 C}$$

is solved first by the method of the variation of constants. For this purpose (3 C) is differentiated with respect to r under the assumption that y_0 is a certain function $c(r)$ of r. In this way

$$y' = Y'c + Yc'\,.$$

From the expression above, together with (5 C) and (6 C), the explicit expression below is obtained for c as a function of r:

$$c(r) = y_0 + \int_{r_0}^{r} Y^{-1}(s)f(s)ds\,. \tag{7 C}$$

Consequently (5 C) takes the form

$$y(r) = Yy_0 + \int_{r_0}^{r} Y(r)Y^{-1}(s)f(s)ds\,. \tag{8 C}$$

Thereafter substitution of (8 C) into the boundary conditions of (1 C) gives for y_0 the expression

$$y_0 = -\int_{r_0}^{a} D^{-1} W_a Y_a Y^{-1}(s) f(s) ds \, , \tag{9 C}$$

where D^{-1} is the inverse of the boundary matrix

$$D = W_0 + W_a Y_a \, . \tag{10 C}$$

Thus (8 C) and (9 C) together give

$$y = -\int_{r_0}^{a} Y(r) D^{-1} W_a Y_a Y^{-1}(s) f(s) ds + \int_{r_0}^{a} Y(r) Y^{-1}(s) f(s) ds \, . \tag{11 C}$$

The final form (11 C) is obtained after some manipulation and use of the identity

$$DD^{-1} = I \, .$$

Thus

$$y(r) = \int_{r_0}^{a} G(r\, , s) f(s) ds \, , \tag{12 C}$$

where

$$G(r\, , s) = \begin{cases} Y(r) D^{-1} W_0 Y^{-1}(s) \, ; & r \geq s \\ - \, Y(r) D^{-1} W_a Y_a Y^{-1}(s) \, ; & r \leq s \end{cases} \tag{13 C}$$

is the Green's matrix of the problem (1 C).

Some results concerning the fundamental matrix of the adjoint boundary value problem: From (152) together with (5 C) it may be readily verified that the fundamental matrix of the adjoint homogeneous boundary value problem (152) satisfies the initial value problem

$$\frac{d\overset{*}{Z}}{dr} = - \overset{*}{Z} A \, ; \; \overset{*}{Z}_0 = I \, . \tag{14 C}$$

From (5 C) and (14 C) one may verify the simple relation

$$\frac{d(\overset{*}{Z}Y)}{dr} = 0 \tag{15 C}$$

between the fundamental matrices Y and $\overset{*}{Z}$. To get the relation (15 C) in a more suitable form, it is integrated from r_0 to a. Thus

$$\overset{*}{Z}Y = I\,. \tag{16 C}$$

Using (16 C) the Green's matrix (13 C) can be represented in its final form, which is

$$G(r\,,\,s) = \begin{cases} Y(r)D^{-1}W_0\overset{*}{Z}(s)\,; & r \leq s \\ -\,Y(r)D^{-1}W_aY_a\overset{*}{Z}(s)\,; & r \leq s \end{cases}\,. \tag{17 C}$$

An integral equation for Y_n: Equation (151) may be stated in the form

$$\frac{dy_n}{dr} = Ay_n + (A_n - A)y_n\,;\; W_0y_{n0} + W_ay_{na} = 0\,, \tag{18 C}$$

where the subscript n relates to the eigenvalue p_n. With the aid of the Green's matrix (17 C) it may be verified that (18 C) is equivalent to the integral equation

$$y_n(r) = \int_{r_0}^{a} G(r\,,\,s)(A_n(s) - A(s))y_n(s)ds\,. \tag{19 C}$$

Since $y_n = Y_ny_0$, it follows from (19 C) that the fundamental matrix Y_n satisfies the integral equation

$$Y_n(r) = \int_{r_0}^{a} G(r\,,\,s)(A_n(s) - A(s))Y_n(s)ds\,. \tag{20 C}$$

Focus mechanism expressed by jump relations: This problem is stated by (148) and (151) as follows:

$$\frac{dy}{dr} = Ay\,;\; W_0y_0 + W_ay_a = 0\,,\; [y] = \underset{s}{y} \text{ at } r = r_s\,. \tag{21 C}$$

For the solution of (21 C) two fundamental matrices are used: the first is $Y^-(r)$ below the focus line $r = r_s$ and satisfies the initial condition $Y^-(r_0) = I$ at $r = r_0$, while the other fundamental matrix

$Y^+(r)$ is selected above focus line so as to satisfy the initial condition $Y^+(r_s) = I$ at $r = r_s$. Since Y^- and Y^+ are fundamental matrices of the same differential equation, they ought to be connected by

$$Y^+ = Y^- C \,, \tag{22 C}$$

from which the constant C is determined by the condition $Y_s^+ = I$ to be

$$C = (Y_s^-)^{-1} \,. \tag{23 C}$$

Since

$$y_s^- = Y_s^- y_0^- \,, \tag{24 C}$$

it follows that

$$y_0^- = (Y_s^-)^{-1} y_s^- \,. \tag{25 C}$$

At the free surface the solution is

$$y_a^+ = Y_a^+ y_s^+ = Y_a^+ y_s^- + Y_a^+ \underset{s}{y} \,, \tag{26 C}$$

where the jump relations from (21 C) have been used.

The boundary conditions of (21 C) can now, with the aid of (22 C), (23 C), (25 C), and (26 C), be expressed as

$$W_0 y_0^- + W_a y_a^+ = W_0 y_0^- + W_a Y_a^- y_0^- + W_a Y_a^- (Y_s^-)^{-1} \underset{s}{y} = 0 \,, \tag{27 C}$$

from which

$$y_0^- = - D^{-1} W_a Y_a^- (Y_s^-)^{-1} \underset{s}{y} \,, \tag{28 C}$$

where

$$D = W_0 + W_a Y_a^- \,.$$

Consequently the solution below the focus line is obtained as

$$y^-(r) = - Y^-(r) D^{-1} W_a Y_a^- (Y^-(s))^{-1} \underset{s}{y} \,. \tag{29 C}$$

For the solution above the focus line, it is found that

$$y^+(r) = Y^+(r)\, y_s^- + Y^+(r) \underset{s}{y} \,, \tag{30 C}$$

from which, after some manipulation and use of (22 C), (23 C), (24 C) and (28 C), it follows that

$$y^+(r) = Y^-(r)D^{-1}W_0(Y^-(s))^{-1}\underset{s}{y} . \qquad (31\ \mathrm{C})$$

Consequently the Green's matrix of (21 C) is, according to (29 C) and (31 C), given by

$$G(r\,,\,s) = \begin{cases} Y^-(r)D^{-1}W_0\overset{*}{Z}{}^-(s)\,; & r \geq s \\ -\,Y^-(r)D^{-1}W_aY_a^-\overset{*}{Z}{}^-(s)\,; & r \leq s \end{cases} , \qquad (32\ \mathrm{C})$$

and accordingly the solution of (21 C) is expressed compactly as

$$y(r) = G(r\,,\,s)\,\underset{s}{y} . \qquad (33\ \mathrm{C})$$