OSCILLATIONS OF AN ISOTROPIC, THERMOVISCOELASTIC, RADIALLY INHOMOGENEOUS SPHERE

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Abstract

In this study, there is presented a model of a self-gravitating, isotropic, thermoviscoelastic, radially inhomogeneous sphere, together with a method for numerical determination of the spectral characteristics of vibrations of such a system.

Introduction

A. E. H. Love [4], H. Takeuchi [6] and [7], Z. Alterman, H. Jarosch and C. L. Pekeris [1], among others, have presented a model earth consisting of an isotropic, elastic, self-gravitating, inhomogeneous sphere with a hydrostatic natural state of stress superposed by an infinitesimal deformation. In this study, a generalization of this model has been suggested. It differs from the previous one mainly in two respects. Firstly, the superposed stress-strain relation is linearly viscoelastic (of the rate or of the functional type, A. C. Eringen [3]). Secondly, the coupling between the mechanical and thermodynamical phenomena has been taken into account.

The model

In the spatial coordinate system, the Fourier-transformed equations of motion which govern the oscillations of the thermoviscoelastic model take the following from:

$$\varrho(x) = \overset{*}{\varrho} - \overset{*}{\varrho}\overline{\nabla} \cdot \bar{u} - \bar{u} \cdot \overline{\nabla}\overset{*}{\varrho} = \overset{*}{\varrho} + \varrho' \tag{1}$$

$$\bar{t}_{\text{tot}} = \overset{*}{p}\bar{g} - \bar{u} \cdot \overline{\nabla}\overset{*}{p}\bar{g} + \bar{t} \tag{2}$$

$$\bar{t} = -\beta T\bar{g} + \lambda(\omega)\overline{\nabla} \cdot \bar{u}\bar{g} + \mu(\omega)[\overline{\nabla}\bar{u} + (\overline{\nabla}\bar{u})^T] \tag{3}$$

$$\overline{\nabla} \cdot \bar{t}_{\text{tot}} + \overset{*}{\varrho}\bar{f}' + \varrho'\bar{f} + \overset{*}{\varrho}\omega^2\bar{u} = 0 \tag{4}$$

$$\bar{q} = \varkappa\overline{\nabla}T \tag{5}$$

$$\overline{\nabla} \cdot \bar{q} + i\omega\overset{*}{\varrho}\gamma T + i\omega\beta T_0\overline{\nabla} \cdot \bar{u} = 0 \tag{6}$$

$$\nabla^2\Phi = -4\pi G\overset{*}{\varrho} \tag{7}$$

$$\nabla^2\Phi' = 4\pi G(\overset{*}{\varrho}\overline{\nabla} \cdot \bar{u} + \bar{u} \cdot \overline{\nabla}\overset{*}{\varrho}) \tag{8}$$

$$\bar{f} = \overline{\nabla}\Phi\,, \bar{f}' = \overline{\nabla}\Phi' \tag{9}$$

For an explanation of the equations (1)—(9) the following, rather compact list has been given:

A. A vector notation has been used, e.g. L. I. Sedov [5].

B. All the equations have been linearized to the first order with respect to $\bar{u}$, $\bar{f}'$, T and their derivatives.

C. (1) is the equation of continuity, where
- $\bar{x}$ is a point in the spatial space
- $\varrho(x)$ is the density of the perturbed body
- $\overset{*}{\varrho}(x)$ is the initial density
- $\overline{\nabla} = g^i\partial_i$ is the nabla operator
- $\bar{g}^i$ is the spatial base system
- $\bar{u} = u^i\bar{g}_i = u_i\bar{g}^i$ is the displacement vector

D. (2) is the total tension in the perturbed body, where
- $\overset{*}{p}$ is the initial hydrostatic pressure component
- $\bar{g} = g^{ij}\bar{g}_i\bar{g}_j = \delta^j_i\bar{g}^i\bar{g}_j = \delta^i_j\bar{g}_i\bar{g}^j = g_{ij}\bar{g}^i\bar{g}^j$ is the metric tensor of the spatial space.

E. (3) is the infinitesimal stress-strain relation, where
- T is the additional infinitesimal absolute temperature due to the perturbation in the deformation
- $\lambda(\omega)$ and $\mu(\omega)$ are the complex elastic moduli, which have been obtained by Fourier-transforming the original stress-strain relations. For the rate-type media:

$$\lambda(\omega) = \frac{P_m S_n - R_n Q_m}{S_n(3R_n + 2S_n)} \quad \text{and} \quad 2\mu(\omega) = \frac{Q_m}{S_n} \tag{10}$$

$$R_n = -\alpha + \sum_{j=1}^{n} \alpha_j(-i\omega)^j\,, \quad S_n = -\delta + \sum_{j=1}^{n} \delta_j(-i\omega)^j$$

$$P_m = \lambda_0 + \sum_{j=1}^{m} \lambda_j(-i\omega)^j \quad \text{and} \quad Q_m = \mu_0 + \sum_{j=1}^{m} \mu_j(-i\omega)^j \tag{11}$$

For the functional type of media:

$$\lambda(\omega) = \lambda_0 + \overset{*}{\lambda}(\omega) \quad \text{and} \quad \mu(\omega) = \mu_0 + \overset{*}{\mu}(\omega).\ \text{Where} \tag{12}$$

$\overset{*}{\lambda}(\omega)$ and $\overset{*}{\mu}(\omega)$ are Fourier-transforms of some weighting functions.

— β, α, α_j, δ, δ_j, λ_0, λ_j, μ_0 and μ_j are thermal and elastic parameters depending on both $\bar{x}$ and the initial state of the body

— $(\overline{\nabla}\bar{u})^T$ is the transpose of $\overline{\nabla}\bar{u}$

F. (4) is the Cauchy equation of motion, where

— $\bar{f}$ is the body force in the unperturbed body and $\bar{f}'$ the additional body force caused by the perturbation

— ω is the angular frequency

G. (5) is the equation of heat conduction, where

— $\bar{q}$ is the vector of heat flux

— $\varkappa$ is the coefficient of thermal conductivity, which depends not only on $\bar{x}$ but also on the initial state of the body

H. (6) is the equation of energy, where

— $\varkappa$ and β are thermal parameters depending on $\bar{x}$ and the initial state of the body.

— T_0 is the temperature in the initial state of the body

I. (7) and (8) are Poisson equations for determining the gravitational potentials Φ and Φ' caused by $\overset{*}{\varrho}(\bar{x})$ and $\varrho'(\bar{x})$ resp., where

— G is the gravitational constant

J. (9) determines the body forces $\bar{f}$ and $\bar{f}'$.

Spherical coordinate system

In view of the spherical symmetry of the problem, it is natural to express the equations (1)—(9) in the spherical coordinate system of fig. 1., where $\bar{r}^0$ is the unit vector in the radial direction $\bar{a}^1(\bar{a}_1)$ and $\bar{a}^2(\bar{a}_2)$ are the two ortogonal surface base vectors.

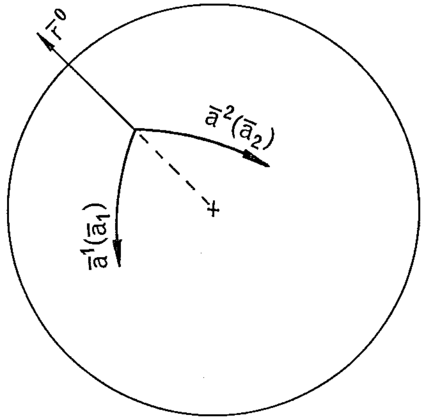

Fig. 1. The used spherical coordinate system

The derivatives of these base vectors will be needed in transformation of the equations (1)—(9) into the spherical coordinate system. Since all the base vectors are independent of r, only derivatives with respect to the surface coordinates $u^\alpha (\alpha = 1,2)$ should be formed. The partial derivatives of $\bar{r}^0$ are

$$\partial_\alpha \bar{r}^0 = \bar{a}_\alpha = a_{\alpha\beta} \bar{a}^\beta \, , \tag{13}$$

where $a_{\alpha\beta}$ are the covariant components of the surface metric tensor. As $\bar{r}^0$, $\bar{a}_1$ and $\bar{a}_2$ form a base system for the space, the vector $\partial_\beta \bar{a}_\alpha$ may be represented with their aid as follows

$$\partial_\beta \bar{a}_\alpha = \Gamma^\varkappa_{\alpha\beta} \bar{a}_\varkappa + b_{\alpha\beta} \bar{r}^0 \tag{14}$$

where $\Gamma^\varkappa_{\alpha\beta} = \partial_\beta \bar{a}_\alpha \cdot \bar{a}^\varkappa$ is the Christoffel symbol for the surface. The coefficient $b_{\alpha\beta}$ is of the form $b_{\alpha\beta} = \partial_\beta \bar{a}_\alpha \cdot \bar{r}^0 = - a_{\alpha\beta} \cdot$ To arrive at this relation, use has been made of the equation $\bar{a}_\alpha \cdot \bar{r}^0 = 0$. After the substitution of $- a_{\alpha\beta}$ for $b_{\alpha\beta}$ in (14), this becomes

$$\partial_\beta \bar{a}_\alpha = \Gamma^\varkappa_{\alpha\beta} \bar{a}_\varkappa - a_{\alpha\beta} \bar{r}^0 \tag{14'}$$

In the same way, the partial derivatives of $\bar{a}^\alpha$ will assume the following form:

$$\partial_\beta \bar{a}^\alpha = - \Gamma^\alpha_{\beta\varkappa} \bar{a}^\varkappa - \delta^\alpha_\beta \bar{r}^0 \tag{15}$$

To extract the spheroidal and torsional parts from equations (1)—(9), use has been made of the Backus representation theorem (G. F.

BACKUS [2]), according to which a surface vector may be expressed with the aid of two scalar potentials. The following expressions have thus been used for the vector and tensor quantities in (1)—(9):

$$\bar{u} = \bar{r}^0 U + \bar{a}_\alpha [a^{\alpha\varkappa} V_{|\varkappa} + \varepsilon^{\alpha\varkappa} W_{|\varkappa}] \tag{16}$$

$$\bar{q} = \bar{r}^0 Q + \bar{a}_\alpha [a^{\alpha\varkappa} M_{|\varkappa} + \varepsilon^{\varkappa\alpha} N_{|\varkappa}] \tag{17}$$

$$\bar{t} = \bar{r}^0 \bar{r}^0 P + (\bar{r}^0 \bar{a}_\alpha + \bar{a}_\alpha \bar{r}^0)[a^{\alpha\varkappa} R_{|\varkappa} + \varepsilon^{\alpha\varkappa} S_{|\varkappa}] + \bar{a}_\alpha \bar{a}_\beta \hat{t}^{\alpha\beta} \tag{18}$$

$$\overline{\nabla} = \bar{r}^0 \partial r + \frac{1}{r} \bar{a}^\varrho \partial_\varrho \tag{19}$$

In (16)—(19) U, V, W, Q, M, N, P, R and S are scalar potentials. $\varepsilon^{\alpha\varkappa}$ is the surface rotator. $V_{|\varkappa}$ denotes the covariant surface derivate of V.

In a spherical coordinate system, more information is obtainable concerning the quantities of (1)—(9) (Z. ALTERMAN *et al.* [1]).

$$\overset{*}{p}\,, r = \overset{**}{\varrho g}\,, \tag{20}$$

where $\overset{*}{g}$ is the acceleration of the gravitational field.

$$\Phi\,, r = - \overset{*}{g} \tag{21}$$

$$\overset{*}{g}\,, r = 4\pi G \overset{*}{\varrho} - \frac{2}{r} \overset{*}{g} \tag{22}$$

$$\Psi = \Phi'\,, r - 4\pi G \overset{*}{\varrho} U \tag{23}$$

Equations for spheroidal and torsional oscillations

If equations (16)—(23) are substituted in (1)—(9) and use is made of both relations (13)—(15), and those which follow:

$$\begin{aligned} &\mathrm{U} = y_1(r)\, S_n\,, \quad V = y_2(r)\, S_n\,, \quad P = y_3(r)\, S_n\,, \\ &\mathrm{R} = y_4(r)\, S_n\,, \quad\quad = y_5(r)\, S_n\,, \quad\quad = y_6(r)\, S_n \\ &\mathrm{T} = y_7(r)\, S_n \ \text{ and } \ Q = y_8(r)\, S_n \end{aligned} \tag{24}$$

for spheroidal oscillations and

$$\mathrm{W} = y_1(r)\, S_n \ \text{ and } \ S = y_2(r)\, S_n \tag{25}$$

for torsional oscillations.

Then, after some manipulation, the following linear homogeneous differential equation system is derived:

$$\frac{dy}{dr} = A(r)y \,, \tag{26}$$

where y is a column vector with the radial parts of (24) and (25) as components. To describe the spheroidal oscillations, $A(r)$ will be a 8x8 matrix, with elements:

$$A_{11} = -\frac{2\lambda}{r(\lambda + 2\mu)} \,, \quad A_{12} = \frac{n(n+1)\,\lambda}{r(\lambda + 2\mu)} \tag{27}$$

$$A_{13} = \frac{1}{\lambda + 2\mu} \,, \quad A_{17} = \frac{\beta}{\lambda + 2\mu}$$

$$A_{14} = A_{15} = A_{16} = A_{18} = 0$$

$$A_{21} = -\frac{1}{r} \,, \quad A_{22} = \frac{1}{r} \,, \quad A_{24} = \frac{1}{\mu}$$

$$A_{23} = A_{25} = A_{26} = A_{27} = A_{28} = 0$$

$$A_{31} = -\overset{*}{\varrho}\omega^2 - 4\,\frac{\overset{**}{\varrho} g}{r} + \frac{4\mu(3\lambda + 2\mu)}{r^2(\lambda + 2\mu)}$$

$$A_{32} = n(n+1)\left[\frac{\overset{**}{\varrho} g}{r} - \frac{2\mu(3\lambda + 2\mu)}{r^2(\lambda + 2\mu)}\right]$$

$$A_{33} = -\frac{4\mu}{r(\lambda + 2\mu)} \,, \quad A_{34} = \frac{n(n+1)}{r}$$

$$A_{36} = -\overset{*}{\varrho} \,, \quad A_{37} = -\frac{4\beta\mu}{r(\lambda + 2\mu)} \,, \quad A_{35} = A_{38} = 0$$

$$A_{41} = \frac{\overset{**}{\varrho} g}{r} - \frac{2\mu(3\lambda + 2\mu)}{r^2(\lambda + 2\mu)} \,, \quad A_{42} = \frac{4n(n+1)\,\mu(\lambda + \mu)}{r^2(\lambda + 2\mu)} - \frac{2\mu}{r^2} - \overset{*}{\varrho}\omega^2$$

$$A_{43} = -\frac{\lambda}{r(\lambda + 2\mu)} \,, \quad A_{44} = -\frac{3}{r} \,, \quad A_{45} = -\frac{\overset{*}{\varrho}}{r}$$

$$A_{47} = \frac{2\beta\mu}{r(\lambda + 2\mu)} \,, \quad A_{46} = A_{48} = 0$$

$$A_{51} = 4\pi G\varrho \,, \; A_{56} = 1 \,, \; A_{52} = A_{53} = A_{54} = A_{55} = A_{56} = A_{57} = 0$$

$$A_{62} = -\frac{4n(n+1)\pi G\overset{*}{\varrho}}{r}, \; A_{65} = \frac{n(n+1)}{r^2}$$

$$A_{66} = -\frac{2}{r}, \; A_{61} = A_{63} = A_{64} = A_{67} = A_{68} = 0$$

$$A_{78} = \frac{1}{\varkappa}, \; A_{71} = A_{72} = A_{73} = A_{74} = A_{75} = A_{76} = A_{77} = 0$$

$$A_{81} = -\frac{4i\omega\beta\mu T_0}{r(\lambda+2\mu)}, \; A_{82} = \frac{2in(n+1)\,\omega\beta\mu T_0}{r(\lambda+2\mu)}$$

$$A_{83} = -\frac{i\omega\beta T_0}{\lambda+2\mu}, \; A_{87} = \frac{n(n+1)}{r} - i\omega\overset{*}{\varrho}\gamma - \frac{i\omega\beta^2 T_0}{\lambda+2\mu}$$

$$A_{88} = -\frac{2}{r}, \; A_{84} = A_{85} = A_{86} = 0$$

To describe the torsional oscillations, $A(r)$ will be a 2x2 matrix with the elements:

$$A_{11} = \frac{1}{r}, \; A_{22} = \frac{1}{\mu} \tag{28}$$

$$A_{21} = -\overset{*}{\varrho}\omega^2 + \frac{\mu}{r^2}(n^2+n-2), \; A_{22} = -\frac{3}{r}$$

The boundary conditions for $y_1, \ldots y_6$ are to be found in Z. ALTERMAN *et al.* [1]. As y_7 is the radial part of the temperature caused by the deformation, it should vanish from a certain depth downwards. Since y_8 is the radial part of the heat conduction vector, it must vanish on the surface of the sphere.

A method for numerical solution of the eigenfrequencies and eigenfunctions of the oscillations

The boundary conditions are expressed symbolically as follows:

$$y(r_0) = y_0 \quad \text{and} \quad L(y(r_n)) = 0 \tag{29}$$

where L is a linear operator, r_0 is the reference depth below which no appreciable amount of energy exists and r refers to the surface of the model sphere. The general solution of (26) is

$$y(r) = R(r, r_0)\, y_0, \tag{30}$$

where the resolvent $R(r, r_0)$ possesses the semi-group property.

$$R(r, r_0) = R(r, r_1) R(r_1, r_0) \tag{31}$$

By use of the property (31), methods can be developed for practical determination of the resolvent. For this purpose, the interval r_0, r_n is divided into n parts r_0, r_1, $r_1, r_2, \ldots$ and r_{n-1}, r_n. After that $y(r_n)$ can be expressed as follows:

$$y(r_n) = R(r_n, r_{n-1}) \ldots R(r_1, r_0)\, y \tag{32}$$

The eigenvalues are obtained by making the $y(r_n)$ in (32) compatible with the boundary condition $L(y(r_n)) = 0$. Thereafter the corresponding eigenfunction is found from (32).

$R(r_{k+1}, r_k)$ is obtained by developing y_{k+1} and y_k into Taylor series around the point $r_{k+\frac{1}{2}} = r_k + \frac{h_k}{2}$, where $h_k = r_{k+1} - r_k$. After the elimination of $y_{k+\frac{1}{2}}$ from these series, it is found that

$$y_{k+1} = \left(I + \frac{h_k}{2} A_{k+\frac{1}{2}}\right)\left(I - \frac{h_k}{2} A_{k+\frac{1}{2}}\right)^{-1} y_k + 0(h_k^3)\,, \tag{33}$$

and from (33)

$$R(r_{k+1}, r_k) = \left(I + \frac{h_k}{2} A_{k+\frac{1}{2}}\right)\left(I - \frac{h_k}{2} A_{k+\frac{1}{2}}\right)^{-1} \tag{34}$$

In (33), I is the unit matrix, $A_{k+\frac{1}{2}}$ is the value of the matrix $A(r)$ at the point $r_{k+\frac{1}{2}}$, and $O(h_k^3)$ is a matrix which behaves under a norm $\|\cdot\|$ like $\|O(h_k^3)\| < Mh_k^3$, where M is a scalar constant.

Some characteristics of the mapping (34) are obtainable by applying it to an eigenvector $u_{k+\frac{1}{2}}$ of the matrix $A_{k+\frac{1}{2}}$. If $\lambda_{k+\frac{1}{2}}$ is an eigenvalue of $A_{k+\frac{1}{2}}$, the corresponding eigenvalue of $R(r_{k+1}, r_k)$ is

$$\mu_{k+\frac{1}{2}} = \frac{1 + \frac{h_k}{2}\lambda_{k+\frac{1}{2}}}{1 - \frac{h_k}{2}\lambda_{k+\frac{1}{2}}} \tag{35}$$

It is found that (35) maps the right half of the $\lambda_{k+\frac{1}{2}}$-plane inside the unit circle of the $\mu_{k+\frac{1}{2}}$-plane. Since the real parts of the eigenvalues of $A_{k+\frac{1}{2}}$ are negative, $R(r_{k+1}, r_k)$ is a contracting mapping.

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